THREE-FOLD DIVISORIAL CONTRACTIONS TO SINGULARITIES OF HIGHER INDICES

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ABSTRACT. We complete the explicit study of a three-fold divisorial contraction whose exceptional divisor contracts to a point, by treating the case where the point downstairs is a singularity of index $n \geq 2$. We prove that if this singularity is of type cA/n then any such contraction is a suitable weighted blow-up; and that if otherwise then the discrepancy is $1/n$ with a few exceptions. Every such exception has an example. Some exceptions allow the discrepancy to be arbitrarily large, but any contraction in this case is described as a weighted blow-up of a singularity of type $cD/2$ embedded into a cyclic quotient of a smooth five-fold. The erratum to the previous paper [14] is attached.

CONTENTS

1. INTRODUCTION

The minimal model program has been formulated to generalise the theory of minimal models of surfaces to higher dimensional varieties, see [18] and [20]. For a given variety with mild singularities, it produces a good variety after a finite sequence of elementary transformations called divisorial contractions and flips. Since Mori completed this program in dimension three by proving the existence of three-fold flips in [25], we have hoped the explicit study of three-folds to strengthen our grasp of the theory of minimal models in higher dimension as well as three-folds themselves. In this paper, we complete the explicit study of a three-fold divisorial contraction whose exceptional divisor contracts to a point, after my previous works [12], [13] and [14] based on the assumption that the target space is Gorenstein. We work in the complex analytic category.

The minimal model program has to work in a category of varieties with mild singularities to avoid problems caused by the existence of small contractions. Reid has introduced the notion of terminal singularities to meet this requirement in [26]. We say that a normal variety X has at worst *terminal singularities* if X is $\mathbb{Q}$-Gorenstein and if every exceptional divisor has positive coefficient in the discrepancy divisor $K_Y - f^*K_X$ for a resolution of singularities $f\colon Y \to X$. Divisorial contractions are defined in the category of varieties with terminal singularities. Let $f\colon Y \to X$ be a proper morphism with connected fibres between normal varieties with at worst terminal singularities. We say that f is an *elementary contraction* if $-K_Y$ is f-ample, and a *divisorial contraction* if moreover the exceptional locus of f is a prime divisor.

We let $f\colon Y \to X$ be a three-fold divisorial contraction. There exist two ways to study f, one starting from Y and the other from X. The former approach is standard in the context of the minimal model program. Mori classified them in the case where Y is smooth in [23], and Cutkosky extended this result to the case where Y is Gorenstein in [9]. The work [23] of Mori has become the foundation of the modern study of three-folds. He proved the cone theorem in original form and applied it to constructing an elementary contraction from a smooth three-fold whose canonical divisor is not nef. On the other hand, the latter approach has turned out to be effective when Y is fairly singular. It is natural to regard f as an extraction from X in view of Sarkisov program, a program which factorises a birational map between Mori fibre spaces, because Sarkisov links of types I and II in this program start from the converse of divisorial contractions, see [5], [7] and [8]. Inspired by Sarkisov program, we have again recognised the importance of the explicit study of three-fold divisorial contractions. In the case where the exceptional divisor E contracts to a curve, f is if exists uniquely determined by the structure of $f(E) \subset Y$, and we can also apply the results [19] and [25] on flipping contractions. Note the results [32] and [33] of Tziolas in this direction. Our object is a divisorial contraction f for which E contracts to a point.

From now on, we let

$$f\colon (Y \supset E) \to (X \ni P)$$

be a germ of a three-fold divisorial contraction whose exceptional divisor E contracts to a point P of index n. The index n is defined as the smallest positive integer such that nK_X is Cartier. We let a/n denote the discrepancy of f, which is defined by the relation $K_Y = f^*K_X + (a/n)E$. Kawamata proved that f is a certain weighted blow-up determined uniquely when P is a terminal quotient singularity in [17], and Corti proved that f is the blow-up when P is an ordinary double point in [6]. I systematically studied f in the case where X is Gorenstein in [14], after the classification results [12] and [13]. I described them when P is a cA point, and bounded the discrepancy by three when P is a cE point. Unfortunately one case has been omitted in [14] when P is a cD point, the correction to which is supplied at the end of this paper. In this paper we treat the case where X is not Gorenstein to complete the study of f, and deduce the following classification theorem.

Theorem 1.1. *Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a three-fold divisorial contraction whose exceptional divisor E contracts to a point P of index $n \geq 2$.*

(i) *If P is of type cA/n (see Section 2 for definition), then there exists a suitable identification*

$$P \in X \cong o \in (x_1 x_2 + g(x_3^n, x_4) = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{n}(1, -1, b, 0)$$

such that f is the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r_1/n, r_2/n, a/n, 1)$ which satisfies the following conditions.
  (a) *$a \equiv br_1$ modulo n and $r_1 + r_2 \equiv 0$ modulo an.*
  (b) *$(a - br_1)/n$ is co-prime to r_1.*
  (c) *g has weighted order $(r_1 + r_2)/n$ with weights $\mathrm{wt}(x_3, x_4) = (a/n, 1)$.*
  (d) *The monomial $x_3^{(r_1+r_2)/a}$ appears in g with non-zero coefficient.*
  Moreover, any such f is a divisorial contraction.

(ii) *If P is of type other than cA/n, then the discrepancy a/n is $1/n$ except for the cases listed in Table 1. Moreover, there exists an example of f in every case in Table 1, see Examples 5.2, 5.6, 5.7, 5.8 and 7.1.*

TABLE 1.

P	a/n	singularities on Y at which E is not Cartier
$cD/2$	1	one point $\frac{1}{2r'}(1, -1, r'+2)$, r' odd, $r' \neq 1$
$cD/2$	1	one point deforming to $2 \times \frac{1}{2r'}(1, -1, r'+1)$, r' even
$cD/2$	2	one point $\frac{1}{2r'}(1, -1, r'+4)$, $r' \equiv \pm 1$ modulo 8, $r' \neq 1$
$cD/2$	$a/2$	two points $\frac{1}{r}(1, -1, \frac{a+r}{2})$, $\frac{1}{r+2}(1, -1, \frac{a+r+2}{2})$, a, r odd
$cD/2$	$a/2$	two points $\frac{1}{r}(1, -1, \frac{a+r}{2})$, $\frac{1}{r+4}(1, -1, \frac{a+r+4}{2})$, $a + r$ even
$cE/2$	1	two points $\frac{1}{6}(1, 5, 5)$, $\frac{1}{2}(1, 1, 1)$

We can consult the works [10] and [11] of Hayakawa when f has discrepancy $1/n$ in the exceptional case (ii) of Theorem 1.1. He studied the divisors with minimal discrepancy over a three-fold non-Gorenstein terminal singularity in detail. On the other hand, the discrepancy a/n is arbitrarily large in the cases in the fourth and fifth lines of Table 1. It is desirable to provide the explicit description of f in these cases.

Theorem 1.2. *Let f be a germ of a three-fold divisorial contraction to a point P of type $cD/2$ with discrepancy $a/2 > 1/2$.*

(i) *If f belongs to the case in the fourth line of Table 1, then there exists a suitable identification of $P \in X$ with*

$$o \in (\phi := x_1^2 + x_1 x_3 q(x_3^2, x_4) + x_2^2 x_4 + \lambda x_2 x_3^{2\alpha-1} + p(x_3^2, x_4) = 0)$$

in the space $\mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{2}(1, 1, 1, 0)$, such that f is the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = ((r+2)/2, r/2, a/2, 1)$ which satisfies the following conditions.
  (a) *a, r are odd and $a \mid r + 1$.*

(b) ϕ has weighted order $r + 1$ with the weights distributed above. Its weighted homogeneous part of weight $r+1$ is irreducible. In fact $x_1 x_3 q$ has weighted order $r + 1$ unless $q = 0$.

(ii) If f belongs to the case in the fifth line of Table 1, then there exists a suitable identification of $P \in X$ with

$$ o \in \left(\begin{array}{c} \phi_1 := x_1^2 + x_2 x_5 + p(x_3^2, x_4) = 0 \\ \phi_2 := x_2 x_4 + x_3^{(r+2)/a} + q(x_3^2, x_4) x_3 x_4 + x_5 = 0 \end{array} \right) $$

in the space $\mathbb{C}^5_{x_1 x_2 x_3 x_4 x_5}/\frac{1}{2}(1,1,1,0,1)$, such that f is the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = ((r+2)/2, r/2, a/2, 1, (r+4)/2)$ which satisfies the following conditions.

(a) $a \mid r + 2$ with $a \neq r + 2$ and $(r+2)/a$ is odd.

(b) ϕ_1 has weighted order $r + 2$ with the weights distributed above. Similarly $q x_3 x_4$ has weighted order $(r + 2)/2$ unless $q = 0$.

Although Theorems 1.1 and 1.2 are the strongest statements, we emphasise that they are intrinsically obtained as corollaries to the conceptual Theorems 1.3 and 1.4 explained below. In contrast to the Gorenstein case, the discrepancy a/n is no longer an integer. The annoying problem is that a and n may have a common divisor g, and consequently there may exist a numerically trivial but linearly non-trivial $\mathbb{Q}$-Cartier divisor $(n/g)K_Y - (a/g)E$. This phenomenon really happens as indicated in Theorem 1.1, but we can still expect the co-primeness of a and n by observing the weights of semi-invariant coordinates of the index-one cover of a three-fold terminal singularity. We prove this co-primeness with the exceptions listed in Table 1 when f is of exceptional type.

Theorem 1.3. *Let $f \colon Y \to X$ be a three-fold divisorial contraction whose exceptional divisor contracts to a point of index n, and a/n the discrepancy of f. If f is of exceptional type (see Section 2 for definition), then a is co-prime to n except for the cases in Table 1.*

The main theorem in [14] is the affirmation of the *general elephant conjecture* for a three-fold divisorial contraction to a Gorenstein point, and the classification is deduced from it. This conjecture has been proposed by Reid in [28]. From the observation of the classification of three-fold terminal singularities by himself in [26] and Mori in [24], he pointed out that a general element in the anti-canonical system, dubbed a *general elephant*, of a three-fold should have at worst Du Val singularities in appropriate situations involving an elementary contraction. The remarkable point is that this approach has settled the problem of the existence of three-fold flips in [25]. We prove that this conjecture holds also for our divisorial contractions. Combining the results in [19] and [25] for the flipping/divisor-to-curve contraction case, we finally obtain the following theorem.

Theorem 1.4. *Let $f \colon Y \to X$ be a germ of a three-fold elementary birational contraction whose central fibre is irreducible. Then a general elephant of Y has at worst Du Val singularities.*

We have positive answers [27], [29] and [31] also for Fano three-folds with mild singularities, whereas there exist counter-examples in [2] when the anti-canonical system has small dimension. Note the result [1] of Alexeev in this direction.

We shall explain how to prove these theorems. Our investigation of f is, as in the case where X is Gorenstein, founded on a classification theorem in the numerical sense, Theorem 2.3, which is obtained by the singular Riemann–Roch formula [28, Theorem 10.2] of Reid and a relative vanishing theorem [18, Theorem 1-2-5] of Kawamata–Viehweg. Theorem 2.3 classifies f into types according to the non-Gorenstein singularities on Y, and we use the dichotomy that f is either of exceptional type or of general type, following this theorem. It is worth while mentioning that Theorem 2.3 does not encode any information on the singularities on Y at which E is Cartier. This is the result of the fact that, because of the locality of f, we need to take the difference along E of Euler–Poincaré characteristics in applying the singular Riemann–Roch formula. Although this lack of information generates larger number of types for f than in the Gorenstein case, it is not difficult to handle most of the exceptional cases.

The method of approaching the co-primeness of a and n is simple. If a and n have a common divisor g, then the index-(n/g) cover $(X' \ni P') \to (X \ni P)$, the covering defined by the $\mathbb{Q}$-Cartier divisor $(n/g)K_X$, is lifted to the crepant covering morphism $(Y' \supset E') \to (Y \supset E)$. The induced morphism $f' : (Y' \supset E') \to (X' \ni P')$ satisfies the conditions to be a divisorial contraction except for the irreducibility of E', and we can apply the numerical classification Theorem 2.3 to f', which narrows down the possibility that a is not co-prime to n. Then, in most cases, Theorem 1.3 is deduced from the numerical analysis which uses the numerically but not linearly trivial divisor $(n/g)K_Y - (a/g)E$. As seen in Corollary 2.5 and Theorem 4.1, it is fairly obvious that for f of exceptional type the co-primeness implies the equality $a = 1$ and the general elephant theorem.

On the other hand, most of the study of f of general type is devoted to the proof of the general elephant Theorem 4.3 in strong form. In the case $a/n > 1$, we prove it by analysing the scheme-theoretic intersection C of E and the birational transform of a general hyperplane section on X, as demonstrated in [14]. This C is a (possibly non-reduced) tree of $\mathbb{P}^1$, and the way of the embedding of C into Y encodes the behaviour of the general elephants of Y. It should be emphasised that we need to keep paying attention to the hidden non-Gorentstein points on Y, the non-Gorenstein points at which E is Cartier, throughout the geometric investigation of Y. In the case $a/n \leq 1$, the general elephant theorem itself is obvious, but the complete proof of Theorem 4.3 involves very complicated division into cases and delicate geometric investigation.

We obtain Theorems 1.1(i) and 1.2 on the explicit description of f by specifying E as a valuation on the function field. Starting with the explicit description of the germ $P \in X$ provided by Mori in [24], we find good coordinates of this germ in the sense that f must be a certain weighted blow-up with weights distributed to these coordinates, thanks to the general elephant theorem. These coordinates are constructed to meet the structure of

the graded ring $\bigoplus_{i\geq 0} f_*\mathcal{O}_Y(-iE)/f_*\mathcal{O}_Y(-(i+1)E)$ in the part of lower degrees, determined by the numerical results from the singular Riemann–Roch formula. We are convinced that once a three-fold singularity is explicitly given, we can classify all the three-fold divisorial contractions to this singularity by combining this method with the singular Riemann–Roch technique and the general elephant theorem. The method of explicit description used for proving Theorem 1.2 is applied to whichever case classified in Theorem 2.3 by formal modification. It is enough to perform such a remaining classification, where the discrepancy is very small, in the course of the application to the global study of three-folds.

The analysis of the cases in Table 1 yields interesting examples of f. In particular for a divisorial contraction f to a point P of type c$D/2$, it is not always true that f is described as a weighted blow-up of a hypersurface in the quotient space $\mathbb{C}^4/\frac{1}{2}(1,1,1,0)$. The description of f involves the identification of $P \in X$ with a cyclic quotient of a complete intersection three-fold in $\mathbb{C}^5$. Such a complication in the c$D/2$ case is recognised also in the study [11] of divisors with minimal discrepancy by Hayakawa.

The structure of this paper is as follows. In Section 2, we deduce the numerical classification Theorem 2.3 and some auxiliary results concerned with this theorem. In Section 3, we discuss the co-primeness and prove Theorem 1.3 except for a few cases to be treated in Section 5. These two

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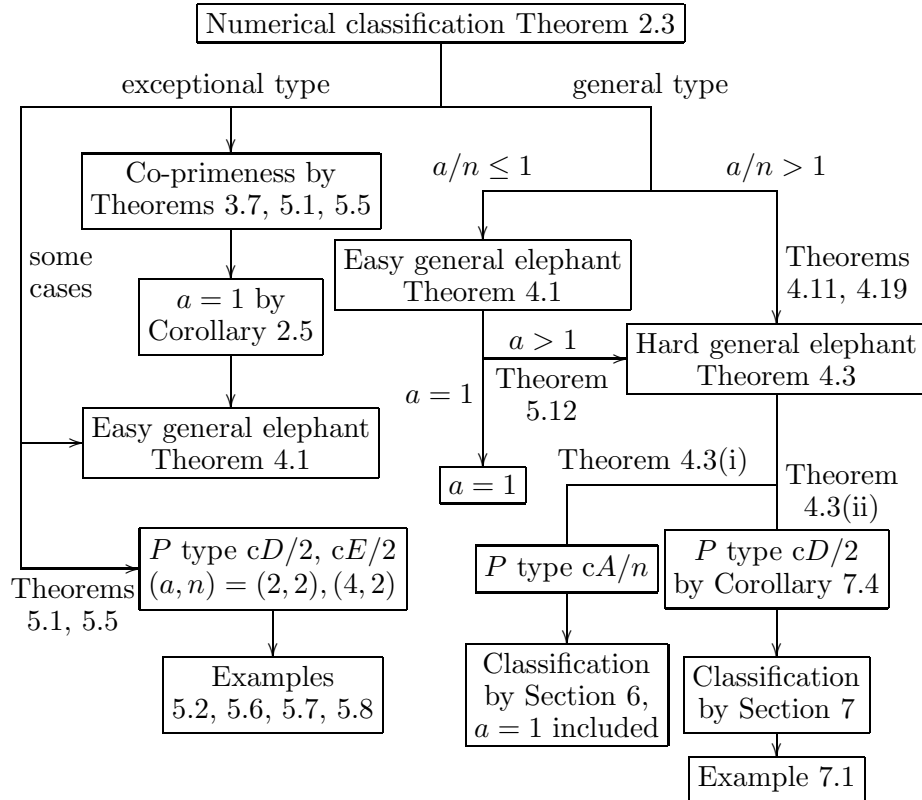
sections are devoted to the numerical analysis, while the rest are devoted to
the geometric analysis. The general elephant theorem is proved in Section
4. Most of this section is allocated to the proof of Theorem 4.3 in the case
$a/n > 1$. To complete Theorems 1.3 and 4.3, we treat some remaining
cases quite explicitly in Section 5. The last two sections are devoted to
the classification of f. In Section 6, we present the explicit description of
divisorial contractions to points of type cA/n. In Section 7, we complete
Theorems 1.1(ii) and 1.2 by studying divisorial contractions of general type
to points of type $cD/2$. Examples for the cases in Table 1 are provided in
Sections 5 and 7. Finally, the erratum to the previous paper [14] is attached
at the end of this paper. All the corrections are made by the methods used
in this paper. The framework of the proof of Theorems 1.1, 1.2, 1.3 and 1.4
is summarised in the flowchart.

2. Numerical results

Our study of three-fold divisorial contractions relies on the classification
of three-fold terminal singularities. We let $P \in X$ be a three-fold germ.
We say that P is a *cDV* (*compound Du Val*) *point* if a general hyperplane
section has at worst a Du Val singularity at P. The singularity P is said
to be cA_n, cD_n, cE_n (*compound* A_n, D_n, E_n) according to the type of the
Du Val singularity on a general hyperplane section. The characterisation of
three-fold Gorenstein terminal singularities is due to Reid.

Theorem 2.1 ([26]). *Let $P \in X$ be a three-fold germ. P is a Gorenstein
terminal singularity if and only if P is an isolated cDV point.*

We then consider a three-fold non-Gorenstein terminal singularity $P \in X$.
Let n denote the index of $P \in X$, that is, the smallest positive integer such
that nK_X is Cartier at P. Take the index-one cover $\pi \colon (X^\sharp \ni P^\sharp) \to
(X \ni P)$, which is a cyclic μ_n-cover. Mori gave a precise description of
this covering, and the classification was completed by Kollár and Shepherd-
Barron in [21, Theorem 6.4].

Theorem 2.2 ([24]). *There exists a μ_n-equivariant identification*

$$P^\sharp \in X^\sharp \cong o \in (\phi = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4},$$

*where x_1, x_2, x_3, x_4 and ϕ are μ_n-semi-invariant. P belongs to one of the
types in Table* 2.

Reid made notable observations on the geometric aspects of a three-fold
terminal singularity $P \in X$ in [28]. He pointed out that a general member of
$|-K_X|$ on a germ of any such singularity has at worst a Du Val singularity at
P, and then proposed the general elephant conjecture that a general elephant
of a three-fold should have at worst Du Val singularities in appropriate
situations involving an elementary contraction. We repeat that a *general
elephant* is by definition a general element in the anti-canonical system. We
let S_X be a general elephant on a germ $P \in X$ of a three-fold non-Gorenstein
terminal singularity. Its pre-image $S_X^\sharp := \pi^{-1}(S_X)_{\mathrm{red}}$ on the index-one cover

TABLE 2.

type	n	ϕ	wt$(x_1, x_2, x_3, x_4; \phi)$
cA/n	n	$x_1 x_2 + g(x_3^n, x_4)$	$(1, -1, b, 0; 0)$ b, n co-prime
c$Ax/2$	2	$x_1^2 + x_4^2 + \phi_{\geq 4}(x_2, x_3)$	$(1, 1, 1, 0; 0)$
c$Ax/4$	4	$x_1^2 + x_2^2 + \phi_{\geq 2}(x_3^2, x_4)$	$(1, 3, 1, 2; 2)$
c$D/2$	2	$x_1^2 + \phi_{\geq 3}(x_2, x_3, x_4)$	$(1, 1, 1, 0; 0)$
c$D/3$	3	$x_4^2 + \phi_{\geq 3}(x_1, x_2, x_3)$	$(1, 2, 1, 0; 0)$
c$E/2$	2	$x_1^2 + x_4^3 + \phi_{\geq 4}(x_2, x_3)$	$(1, 1, 1, 0; 0)$

$P^\sharp \in X^\sharp$ is a $\mathbb{Q}$-Cartier divisor, and satisfies S_2 condition by [20, Corollary 5.25]. Hence the covering $S_X^\sharp \to S_X$ is a crepant morphism between normal surfaces with Du Val singularities. We reproduce the list of this covering in Table 3.

TABLE 3.

type	covering		
cA/n	A_{k-1}	$\to$	A_{kn-1}
c$Ax/2$	A_{2k-1}	$\to$	D_{k+2}
c$Ax/4$	A_{2k-2}	$\to$	D_{2k+1}
c$D/2$	D_{k+1}	$\to$	D_{2k}
c$D/3$	D_4	$\to$	E_6
c$E/2$	E_6	$\to$	E_7

Reid also noted that any three-fold terminal singularity $P \in X$ has a small deformation to a basket of terminal quotient singularities P_i, called *fictitious singularities*, although Mori had already observed it in [24]. This basket plays a pivotal role in the singular Riemann–Roch formula [28, Theorem 10.2] of Reid. If P is of type other than c$Ax/4$, then any local index at P_i equals that at P, and if P is of type c$Ax/4$, then one of P_i has local index 4 and all the others have local index 2.

We return to the study of three-fold divisorial contractions. We let

$$f \colon (Y \supset E) \to (X \ni P)$$

be a germ of a three-fold divisorial contraction whose exceptional divisor E contracts to a point P of index n. We let a/n denote its discrepancy defined by the relation $K_Y = f^* K_X + (a/n)E$. We shall expand the numerical results [12, Section 4] and [14, Section 2] into our general situation. Unlike the case where X is Gorenstein, in general E does not generate the torsion part of the local class group at a point of Y. By this reason, we need to consider both the divisors K_Y and E simultaneously. We set

(2.1) $$D_{i,j} := iK_Y + jE$$

for integers i, j. There exists a natural exact sequence

(2.2) $$0 \to \mathcal{O}_Y(D_{i,j-1}) \to \mathcal{O}_Y(D_{i,j}) \to \mathcal{Q}_{i,j} \to 0.$$

The quotient sheaf $\mathcal{Q}_{i,j}$ is an $\mathcal{O}_E$-module satisfying S_2 condition by [20, Proposition 5.26]. Note that $\mathcal{Q}_{0,0} = \mathcal{O}_E$ and $\mathcal{Q}_{1,1} = \omega_E$. We set

$$d(i, j) := \chi(\mathcal{Q}_{i,j}) = \chi(\mathcal{O}_Y(D_{i,j})) - \chi(\mathcal{O}_Y(D_{i,j-1})).$$

We need to extend f to a morphism of proper varieties by the algebraisation theorems [3] and [4] of Artin when we regard $d(i,j)$ as the difference $\chi(\mathcal{O}_Y(D_{i,j})) - \chi(\mathcal{O}_Y(D_{i,j-1}))$. Note that the value $d(i,j)$ depends only on the class of (i,j) in $\mathbb{Z}^2/\mathbb{Z}(n,-a)$.

We shall compute $d(i,j)$ by the singular Riemann–Roch formula. We let $I_0 := \{Q, \text{ of type } \frac{1}{r_Q}(1,-1,b_Q)\}$ denote the basket of fictitious singularities from singularities on Y, and e_Q for $Q \in I_0$ the smallest positive integer such that $E \sim e_Q K_Y$ at Q. Such an e_Q exists by [15, Corollary 5.2]. Note that $n \equiv ae_Q$ modulo r_Q by the global relation $nK_Y \sim aE$. By replacing b_Q with $r_Q - b_Q$ if necessary, we may assume that $v_Q := \overline{e_Q b_Q} \leq r_Q/2$, where $\bar{}$ denotes the residue modulo r_Q, that is, $\overline{k} := k - \lfloor k/r_Q \rfloor r_Q$. We set $I := \{Q \in I_0 \mid v_Q \neq 0\}$. If a non-Gorenstein point Q on Y is of type other than $cAx/4$, then v_{Q_i} takes the same value for every fictitious singularity Q_i from Q. If Q is of type $cAx/4$, then v_{Q_0} is either $\overline{\pm 1}$, 2 or 0 for the fictitious singularity Q_0 of index 4, and any other v_{Q_i} is respectively 1, 0, 0.

By this notation, the singular Riemann–Roch formula implies that

$$(2.3) \qquad d(i,j) = \frac{1}{12}\left(6\left(\frac{a}{n}i + j\right)^2\right.$$
$$\left. - 6\left(\frac{a}{n}+1\right)\left(\frac{a}{n}i+j\right) + \left(\frac{a}{n}+1\right)\left(\frac{a}{n}+2\right)\right)E^3$$
$$+ \frac{1}{12}E\cdot c_2(Y) + A_{i,j} - A_{i,j-1}.$$

Here the contribution term $A_{i,j}$ is given by $A_{i,j} := \sum_{Q\in I} A_Q(\overline{i+je_Q})$, where

$$A_Q(k) := -k\frac{r_Q^2-1}{12r_Q} + \sum_{l=1}^{k-1}\frac{\overline{lb_Q}(r_Q - \overline{lb_Q})}{2r_Q}.$$

$A_Q(k)$ is determined by k modulo r_Q. We emphasise that the formula (2.3) no longer encodes any information on the fictitious singularities at which E is Cartier. This is never the case when X is Gorenstein, but in general the set I does not necessarily coincide with I_0. We say that a non-Gorenstein point Q on Y is a *hidden non-Gorenstein point* if E is Cartier at Q. Note that the local index at any hidden non-Gorenstein point is a divisor of n by the relation $nK_Y \sim aE$. It is difficult to control the value of $A_Q(k)$ itself, but the following formulae make $A_Q(k)$ more accessible.

$$(2.4) \qquad A_Q(k+1) - A_Q(k) = -\frac{r_Q^2-1}{12r_Q} + B_{r_Q}(kb_Q),$$
$$(2.5) \qquad A_Q(k) + A_Q(-k) = -B_{r_Q}(kb_Q),$$

where $B_r(k) := (\overline{k}\cdot\overline{r-k})/2r$. By (2.3) and (2.4), we obtain that

$$(2.6) \qquad d(i+1,j) - d(i,j) = \left(\frac{a}{n}i + j - \frac{1}{2}\right)\frac{a}{n}E^3 + B_{i,j} - B_{i,j-1},$$

where $B_{i,j} := \sum_{Q\in I} B_{r_Q}(ib_Q + jv_Q)$.

On the other hand, the relative vanishing theorem [18, Theorem 1-2-5] implies that $R^k f_* \mathcal{O}_Y(D_{i,j}) = 0$ for $k \geq 1$ whenever $ia/n + j \leq a/n$. In

particular by (2.2), if $ia/n + j \leq a/n$ then

$$(2.7) \qquad H^k(\mathcal{Q}_{i,j}) = \begin{cases} f_*\mathcal{O}_Y(D_{i,j})/f_*\mathcal{O}_Y(D_{i,j} - E) & \text{for } k = 0, \\ 0 & \text{for } k \geq 1. \end{cases}$$

Therefore

$$(2.8) \qquad d(i,j) = \dim f_*\mathcal{O}_Y(D_{i,j})/f_*\mathcal{O}_Y(D_{i,j} - E) \quad \text{if } ia/n + j \leq a/n.$$

The elements of the $\mathcal{O}_X$-module $f_*\mathcal{O}_Y(D_{i,j})/f_*\mathcal{O}_Y(D_{i,j} - E)$ correspond to the effective divisors on X linearly equivalent to iK_X which have multiplicity $-(ia/n + j)$ along E. In particular,

$$(2.9) \qquad d(i,j) = \begin{cases} 0 & \text{if } 0 \leq ia/n + j \leq a/n \text{ and } (i,j) \notin \mathbb{Z}(n,-a), \\ 1 & \text{if } (i,j) \in \mathbb{Z}(n,-a). \end{cases}$$

We explain (2.9) in the case where $ia/n + j = 0$. Set $g := \gcd(a,n)$ and $a = ga'$, $n = gn'$. Then $D_{n',-a'} = n'f^*K_X$ is numerically trivial and g is the smallest positive integer such that $gD_{n',-a'}$ is Cartier. We can write $(i,j) = k(n', -a')$ by some integer k if $ia/n + j = 0$. Then

$$d(i,j) = \dim \mathcal{O}_X(kn'K_X)/f_*\mathcal{O}_Y(kn'f^*K_X - E),$$

whence $d(i,j) = 1$ if and only if $kn'K_X$ is Cartier at P, or equivalently $(i,j) \in \mathbb{Z}(n,-a)$.

We classify f according to the numerical data $J := \{(r_Q, v_Q)\}_{Q \in I}$.

Theorem 2.3. *f belongs to one of the types in Table 4.*

TABLE 4.

No	J	$(a/n)E^3$	No	J	$(a/n)E^3$
1	$(6,3)$	$1/2$	10	$(2,1), (2,1), (r,1)$	$1/r$
2	$(7,3)$	$2/7$	11	$(2,1), (3,1), (3,1)$	$1/6$
3	$(8,3)$	$1/8$	12	$(2,1), (3,1), (4,1)$	$1/12$
4	$(4,2), (r,1)$	$1/r$	13	$(2,1), (3,1), (5,1)$	$1/30$
5	$(5,2), (2,1)$	$3/10$	14	$(r,2)$	$4/r$
6	$(5,2), (3,1)$	$2/15$	15	$(r_1,1), (r_2,1), r_1 \leq r_2$	$1/r_1 + 1/r_2$
7	$(5,2), (4,1)$	$1/20$	16	$(r,1)$	$1 + 1/r$
8	$(6,2), (2,1)$	$1/6$	17	$\emptyset$	2
9	$(7,2), (2,1)$	$1/14$			

Proof. $d(1,0) - d(0,0) = -1$ by (2.9). Applying (2.6) for $(i,j) = (0,0)$, we have

$$-1 = -\frac{1}{2}\left(\frac{a}{n}E^3\right) - \sum_{Q \in I} \frac{v_Q(r_Q - v_Q)}{2r_Q}.$$

All the possibilities of J satisfying this equality are listed in Table 4.   q.e.d.

We divide No 15 in Table 4 into cases for convenience. We say that f is of No $15'$, No $15''$ if J comes from one, respectively two non-Gorenstein points on Y. No $15'$ is further divided into two cases: No $15'$a if $J = \{(r,1), (r,1)\}$, and No $15'$b if $J = \{(2,1), (4,1)\}$. We say that f is *of exceptional type* if f is of No 1-14, $15'$ or 17, and *of general type* if f is of No $15''$ or 16.

We shall present some auxiliary results based on the singular Riemann–Roch formula and the relative vanishing theorem. We let R denote the least common multiple of all the r_Q for $Q \in I$, and R^* that of all the $r_Q/\gcd(r_Q, v_Q)$ for $Q \in I$.

Lemma 2.4. $R^* E^3$ *and* $(a/n)^2 R E^3$ *are integers.*

Proof. By (2.3), $d(0, R^* + j) - d(0, j) = (j + (R^* - a/n - 1)/2) R^* E^3$. Thus $R^* E^3$ is an integer. By (2.6), $(d(R + i + 1, j) - d(R + i, j)) - (d(i + 1, j) - d(i, j)) = (a/n)^2 R E^3$. Thus $(a/n)^2 R E^3$ is an integer. q.e.d.

The following is an easy corollary to Lemma 2.4.

Corollary 2.5. *Suppose that* f *is of exceptional type with index* $n \geq 2$. *If* a *is co-prime to* n, *then* $a = 1$.

Proof. For instance, we consider the case No $15'$a. Then $R^* E^3 = 2n/a$ and $(a/n)^2 R E^3 = 2a/n$. Hence $a = 1$ by the assumption and Lemma 2.4. q.e.d.

We shall consider a non-trivial effective divisor $L_{i,j}$ on Y which is linearly equivalent to $D_{i,j}$ and intersects E properly. Note that such an $L_{i,j}$ exists if and only if $f_* \mathcal{O}_Y(D_{i,j} - E) \neq f_* \mathcal{O}_Y(D_{i,j})$ and $D_{i,j} \not\sim 0$. In particular, the assumption that $ia/n + j < 0$ is implicitly imposed. By the following exact sequences, we construct an $\mathcal{O}_{L_{i,j} \cap E}$-module $\mathcal{R}_{i,j}^{i',j'}$ which is naturally isomorphic to $\mathcal{O}_Y(D_{i',j'}) \otimes \mathcal{O}_{L_{i,j} \cap E}$ outside finitely many points.

$$(2.10) \qquad 0 \to \mathcal{O}_Y(D_{i',j'} - E) \to \mathcal{O}_Y(D_{i',j'}) \to \mathcal{Q}_{i',j'} \to 0,$$

$$(2.11) \qquad 0 \to \mathcal{Q}_{i'-i,j'-j} \to \mathcal{Q}_{i',j'} \to \mathcal{R}_{i,j}^{i',j'} \to 0,$$

$$(2.12) \quad
\begin{array}{ccccc}
 & & \mathcal{O}_Y(D_{i',j'}) & & \\
 & & \downarrow & & \\
0 \to (\text{finite length}) \to & \mathcal{O}_Y(D_{i',j'}) \otimes \mathcal{O}_E & \to & \mathcal{Q}_{i',j'} & \to 0 \\
 & \downarrow & & \downarrow & \\
0 \to (\text{finite length}) \to & \mathcal{O}_Y(D_{i',j'}) \otimes \mathcal{O}_{L_{i,j} \cap E} & \to & \mathcal{R}_{i,j}^{i',j'} & \to 0.
\end{array}$$

The map $\mathcal{O}_Y(-L_{i,j}) \otimes \mathcal{O}_E \to \mathcal{Q}_{-i,-j}$ is surjective because it factors the surjective map $\mathcal{O}_Y(-L_{i,j}) \twoheadrightarrow \mathcal{Q}_{-i,-j}$. Hence $\mathcal{Q}_{-i,-j}$ is the ideal sheaf of $L_{i,j} \cap E \subset E$, and thus $\mathcal{R}_{i,j}^{0,0} = \mathcal{O}_{L_{i,j} \cap E}$.

Lemma 2.6. (i) $f_* \mathcal{O}_Y(D_{i,j}) \twoheadrightarrow H^0(\mathcal{Q}_{i,j})$ *for any* i, j.

$$h^1(\mathcal{Q}_{i,j}) = 0 \quad \text{unless } a/n < ia/n + j < 1.$$

$$h^2(\mathcal{Q}_{i,j}) = \begin{cases} 0 & \text{if } ia/n + j \leq a/n, \\ d(1 - i, 1 - j) & \text{if } ia/n + j \geq 1. \end{cases}$$

(ii) *Consider* $L_{i,j}$ *as above. Then,*

$$H^0(\mathcal{Q}_{i',j'}) \twoheadrightarrow H^0(\mathcal{R}_{i,j}^{i',j'}) \text{ unless } a/n < (i' - i)a/n + (j' - j) < 1.$$

$$h^1(\mathcal{R}_{i,j}^{i',j'}) = \begin{cases} 0 & \text{if } (i' - i)a/n + (j' - j) \leq a/n + 1 \\ & \quad \text{and } (i' - i - 1, j' - j - 1) \notin \mathbb{Z}(n, -a), \\ 1 & \text{if } (i' - i - 1, j' - j - 1) \in \mathbb{Z}(n, -a) \end{cases}$$

unless $a/n < i'a/n + j' < 1$.

Proof. The lemma is derived from the Grothendieck duality $H^k(\mathcal{Q}_{i,j}) = H^{2-k}(\mathcal{Q}_{1-i,1-j})$ and (2.7), (2.8), (2.9), (2.11).   q.e.d.

Corollary 2.7. *Suppose that $-(ia/n + j) \leq a/n + 1$ with $(i+1, j+1) \notin \mathbb{Z}(n, -a)$ and that the estimate $a/n < -(ia/n + j) < 1$ does not hold. Then $h^0(\mathcal{O}_{L_{i,j} \cap E}) = 1$ and $h^1(\mathcal{O}_{L_{i,j} \cap E}) = 0$. In particular, $L_{i,j} \cap E$ has no embedded points and $(L_{i,j} \cap E)_{\mathrm{red}}$ is a tree of $\mathbb{P}^1$.*

The way of the embedding of $L_{i,j} \cap E$ into Y encodes the behaviour of the general elephants of Y. According to Corollary 2.7, the fundamental ingredients we should study are smooth curves on Y. We let C be a smooth curve on Y, and Q a quotient singularity $\mathbb{C}^3_{x_1 x_2 x_3}/\frac{1}{r_Q}(1, -1, b_Q)$ on Y which C passes through. We shall review the local study of the germ $Q \in C \subset Y$ in [14, Section 2] after [25, Section 2].

We take the index-one cover $\pi \colon (Y^\sharp \ni Q^\sharp) \to (Y \ni Q)$ and set $C^\sharp := (C \times_Y Y^\sharp)_{\mathrm{red}}$. Let n_Q^C denote the number of the irreducible components of $C^\sharp$, and $Q^\dagger \in C^\dagger$ the normalisation of one of the irreducible components of $C^\sharp$. We let $t \in \mathcal{O}_{C,Q}$ and $t^{n_Q^C/r_Q} \in \mathcal{O}_{C^\dagger,Q^\dagger}$ be uniformising parameters of C and $C^\dagger$. Let a_i denote the smallest positive integer such that there exists a semi-invariant function with weight $\mathrm{wt}\, x_i$ whose image in $\mathcal{O}_{C^\dagger,Q^\dagger}$ has order a_i/r_Q with respect to t. Then the coordinates of $Y^\sharp$ can be taken so that $x_i|_{C^\dagger} = t^{a_i/r_Q}$ for $i = 1, 2, 3$. Such a description is called a *normal form* of $Q \in C \subset Y$. Note that

$$(a_1, a_2, a_3) \in \mathbb{Z} \cdot n_Q^C(1, -1, b_Q) + r_Q \mathbb{Z}^3.$$

For $l \in \mathbb{Z}/(r_Q)$, we let $w_Q^C(l)$ denote the smallest non-negative integer such that $(w_Q^C(l), l) \in \mathbb{Z} \times \mathbb{Z}/(r_Q)$ is contained in the semi-group

$$(2.13) \qquad G_Q^C := \mathbb{Z}_{\geq 0}(a_1, 1) + \mathbb{Z}_{\geq 0}(a_2, -1) + \mathbb{Z}_{\geq 0}(a_3, b_Q) \subset \mathbb{Z} \times \mathbb{Z}/(r_Q).$$

We note that $(w, l) \in G_Q^C$ means the existence of a semi-invariant function with weight l whose restriction to $C^\dagger$ is t^{w/r_Q}. In particular $w_Q^C(0) = 0$ and $(r_Q, 0) \in G_Q^C$.

Let $\mathcal{F}$ denote the reflexive sheaf on the germ $Q \in Y$ which is isomorphic to the ideal sheaf defined by $x_1 = 0$ outside Q. Note that $\mathcal{F}^{[\otimes v_Q]} \cong \mathcal{O}_Y(E)$ and $\mathcal{F}^{[\otimes b_Q]} \cong \mathcal{O}_Y(K_Y)$, where $\mathcal{F}^{[\otimes i]}$ denotes the double dual of $\mathcal{F}^{\otimes i}$. Then for arbitrary integers $j_1, \ldots, j_k$ with $\sum_{1 \leq i \leq k} j_i = 0$,

$$(2.14) \quad \mathrm{Im}\Big(\mathcal{F}^{[\otimes j_1]} \otimes \cdots \otimes \mathcal{F}^{[\otimes j_k]} \otimes \mathcal{O}_C \to \mathcal{O}_C\Big)$$
$$= \mathcal{O}_C\Big(-\big(\sum_{1 \leq i \leq k} w_Q^C(-j_i)/r_Q\big) \cdot Q\Big).$$

3. Co-primeness

In this section we discuss the co-primeness of a and n and prove Theorem 1.3 except for a few cases to be treated in Section 5. We start with listing the types of f for which a and n may have a non-trivial common divisor.

Let p be a common divisor of a and n, and set $a = pa_p$, $n = pn_p$. We take the index-n_p cover $\alpha_X \colon (X' \ni P') \to (X \ni P)$, the covering defined by

$n_p K_X$, and lift it over Y by taking the normalisation of Y in the function field of X'. By this procedure, we obtain the commutative diagram

$$(3.1) \qquad \begin{array}{ccc} (Y' \supset E') & \xrightarrow{\ \alpha_Y\ } & (Y \supset E) \\ f' \downarrow & & \downarrow f \\ (X' \ni P') & \xrightarrow{\ \alpha_X\ } & (X \ni P), \end{array}$$

where $E' := \alpha_Y^* E$. The morphism α_Y is just the covering defined by $n_p f^* K_X = n_p K_Y - a_p E$. In particular α_Y is étale outside finitely many points, which implies that $K_Y' = \alpha_Y^* K_Y$ and that E' is an f'-anti-ample reduced divisor. Hence Y' has also at worst terminal singularities, and f' satisfies the conditions to be a divisorial contraction except for the irreducibility of E'.

Now we should notice that all the results in Section 2 hold without the condition of the irreducibility of E, in contrast to the use of the reducedness of E to deduce that $d(0,0) = 1$. By applying Theorem 2.3 to both f and f', we can restrict the possibilities of the covering (3.1). It is worth while mentioning that, although we can construct the covering (3.1) by using $(n/q)K_X$ for any divisor q of n, the morphism α_Y is no longer étale at the generic point of E unless q is also a divisor of a. Once α_Y ramifies at the generic point of E, there is no guarantee that Y' has terminal singularities, and we can not apply the singular Riemann–Roch formula to Y'.

We return to our situation. The covering (3.1) is a cyclic μ_p-cover for a common divisor p of a and n. In order to compare the numerical data of f and f', we examine the variations of the basket of fictitious singularities and the value of $(a/n)E^3$ by this covering. It is obvious that

$$(3.2) \qquad \frac{a_p}{n_p} E'^3 = p \cdot \frac{a}{n} E^3.$$

We shall investigate the variation of the basket. Since the covering (3.1) is decomposed into coverings of prime degrees, we assume that p is a prime divisor for the time being. We let Q be a singularity of Y. There exist two cases of the local behaviour of α_Y at Q. If α_Y is étale at Q, then each fictitious singularity Q_i of type $\frac{1}{r_{Q_i}}(1, -1, b_{Q_i})$ from Q is divided into p fictitious singularities of type $\frac{1}{r_{Q_i}}(1, -1, b_{Q_i})$ on Y'. If α_Y is ramified at Q, then $p \mid r_Q$ and each Q_i varies into a fictitious singularity of type $\frac{1}{r_{Q_i}/p}(1, -1, b_{Q_i})$ on Y', provided that Q is of type other than $\mathrm{c}Ax/4$. If Q is of type $\mathrm{c}Ax/4$, then $p = 2$ and the basket $\{\frac{1}{4}(1, -1, \pm 1), s \times \frac{1}{2}(1, 1, 1)\}$ varies into $\{(2s + 1) \times \frac{1}{2}(1, 1, 1)\}$. On the other hand, the value e_Q is preserved modulo index since $K_Y' = \alpha_Y^* K_Y$ and $E' = \alpha_Y^* E$. Therefore we have obtained the following lemma on $J = \{(r_Q, v_Q)\}_{Q \in I}$.

Lemma 3.1. *Suppose that p is a common prime divisor of a and n. Let $Q \in I_0$ be a fictitious singularity of Y equipped with the data (r_Q, v_Q).*

- (i) *If α_Y is étale at Q, then there exist exactly p fictitious singularities Q' of Y' above Q. Each such Q' is equipped with the data $(r_{Q'}, v_{Q'}) = (r_Q, v_Q)$.*

(ii) *If α_Y is ramified at Q, then there exists exactly one fictitious sin-gularity Q' of Y' above Q. Q' is equipped with the data $(r_{Q'}, v_{Q'}) = (r_Q/p, \overline{\pm v_Q})$, where $\overline{\pm v_Q}$ is the minimum of $\overline{v_Q}$ and $\overline{-v_Q}$.*

We list all the possibilities of α_Y in terms of J by Theorem 2.3, (3.2) and Lemma 3.1.

Theorem 3.2. *Suppose that p is a common prime divisor of a and n. Let $J' := \{(r_Q', v_Q')\}_{Q' \in I'}$ be the data of Y' constructed in the same fashion as J. Then α_Y belongs to one of the cases in Table 5.*

TABLE 5.

No of f	J	p	J'
1	$(6,3)$	3	$(2,1)$
2	$(7,3)$	7	$\emptyset$
4	$(4,2)$, $(r,1)$	2	$(r,1)$, $(r,1)$
8	$(6,2)$, $(2,1)$	2	$(3,1)$, $(2,1)$, $(2,1)$
		3	$(2,1)$, $(2,1)$, $(2,1)$
10	$(2,1)$, $(2,1)$, $(r,1)$	2	$(2,1)$, $(2,1)$, $(r/2,1)$
		2	$(r,1)$, $(r,1)$
11	$(2,1)$, $(3,1)$, $(3,1)$	3	$(2,1)$, $(2,1)$, $(2,1)$
12	$(2,1)$, $(3,1)$, $(4,1)$	2	$(3,1)$, $(3,1)$, $(2,1)$
14	$(r,2)$	p	$(r/p,2)$, $r/p \geq 2$
$15'$a, $15''$	$(r_1,1)$, $(r_2,1)$	p	$(r_1/p,1)$, $(r_2/p,1)$

Proof. For instance, we consider the case No 1. Then $(a/n)E^3 = 1/2$. Let Q be the fictitious singularity of Y such that $(r_Q, v_Q) = (6,3)$. Because J' contains at most one $(6,3)$ by Table 4, α_Y must be ramified at Q by Lemma 3.1, whence $p = 2$ or 3. If $p = 2$ then $J' = \emptyset$ and $(a_p/n_p)E'^3 = 1$ by (3.2) and Lemma 3.1, but it is impossible by the list of the possible types of f' in Table 4. If $p = 3$ then $J' = \{(2,1)\}$ and $(a_p/n_p)E'^3 = 3/2$. In this case f' is of No 16.   q.e.d.

If a is not co-prime to n, then in particular $a \neq 1$. The following numerical lemma is derived from the results [16] and [22] on minimal discrepancies by Kawamata in the non-Gorenstein case and by Markushevich in the Goren-stein case.

Lemma 3.3. *Suppose that X is singular at P and that $a \neq 1$. Then there exist a singularity $Q \in Y$ of index r_Q at which $E \sim e_Q K_Y$ and positive integers n_m, n_a satisfying the following conditions.*

    (i) *$an_m + nn_a = r_Q$.*
    (ii) *$n_m + e_Q n_a \equiv 0$ modulo r_Q.*
In particular, $a + n \leq r_Q$.

Proof. The results [16] and [22] by Kawamata and Markushevich are the following. For a three-fold terminal singularity $P \in X$ of index n, there exists an exceptional divisor above P whose discrepancy with respect to K_X is $1/n$ unless X is smooth at P. Hence if $a \neq 1$ then there exist a birational morphism $g: Z \to Y$ from a smooth variety Z and a g-exceptional prime

divisor F such that $K_Z = (f \circ g)^* K_X + (1/n)F + (\text{others})$. $g(F)$ has to be a non-Gorenstein point Q of index say r_Q. Let e_Q be an integer such that $E \sim e_Q K_Y$ at Q. Set

$$K_Z = g^* K_Y + (n_a/r_Q)F + (\text{others}),$$
$$g^* E = g_*^{-1} E + (n_m/r_Q)F + (\text{others}).$$

Then the equalities (i) and (ii) hold.   q.e.d.

Corollary 3.4.   (i) *a is co-prime to n unless f is of No 4, 8, 10, 12, 14 15′a or 15″.*
   (ii) *If f is of No 8 or 12, then a is co-prime to n or $(a, n) = (2, 2)$.*

Proof. For (i), it suffices to exclude the cases No 1, 2 and 11 from Table 5. For instance, we consider the case No 1. a and n are divided by 3, and Q in Lemma 3.3 must be equipped with the data $(r_Q, v_Q) = (6, 3)$ and $e_Q = 3$. Thus $a n_m/r_Q \geq 3/2$ by (ii) in Lemma 3.3, but this contradicts (i) in Lemma 3.3. (ii) is proved in the same way.   q.e.d.

For a common divisor p of a and n, we set

$$d_Q^p := \overline{n_p - e_Q a_p}.$$

Then $p d_Q^p \equiv 0$ modulo r_Q by $n K_Y \sim a E$, and α_Y is locally the covering defined by the divisor $n_p f^* K_X = n_p K_Y - a_p E \sim d_Q^p K_Y$ at Q. In Lemma 3.1, (i) occurs when $d_Q^p \equiv 0$ modulo r_Q, while (ii) occurs when $p d_Q^p \equiv 0$ but $d_Q^p \not\equiv 0$ modulo r_Q. We obtain Table 6 which lists all the possible coverings of prime degrees, based on Lemma 3.1, Theorem 3.2 and Corollary 3.4.

TABLE 6.

No of f	p	$Q_i \in I$ with (r_{Q_i}, v_{Q_i})	$d_i := d_{Q_i}^p$
4	2	Q_1: $(4, 2)$, Q_2: $(r, 1)$	$(d_1, d_2) = (2, 0)$
8	2	Q_1: $(6, 2)$, Q_2: $(2, 1)$	$(d_1, d_2) = (3, 0)$
10	2	Q_1: $(2, 1)$, Q_2: $(2, 1)$, Q_3: $(r, 1)$	$(d_1, d_2, d_3) = (0, 1, r/2)$
	2		$(d_1, d_2, d_3) = (1, 1, 0)$
12	2	Q_1: $(2, 1)$, Q_2: $(3, 1)$, Q_3: $(4, 1)$	$(d_1, d_2, d_3) = (1, 0, 2)$
14	p	Q: $(r, 2)$	
15′a, 15″	p	Q_1: $(r_1, 1)$, Q_2: $(r_2, 1)$	

We examine No 14 and 15 without the assumption that p is a prime. Recall the notation that $g = \gcd(a, n)$, $a = g a'$, $n = g n'$.

Lemma 3.5.   (i) *If f is of No 14 or 15′a with $g \geq 2$, then $g = 2$ and $d_Q^2 = r_Q/2$ for any $Q \in I$.*
   (ii) *If f is of No 15″ with $g \geq 2$, then $\overline{d_{Q_1}^g b_1}/r_1 + \overline{d_{Q_2}^g b_2}/r_2 = 1$.*

Proof. By (2.6), we have

$$(d(n' + 1, -a') - d(n', -a')) - (d(1, 0) - d(0, 0))$$
$$= (B_{n', -a'} - B_{n', -a'-1}) + B_{0, -1}.$$

Suppose that $g \geq 2$. By (2.9) the left-hand side is 1, whence

$$(3.3) \qquad 1 = \sum_{Q \in I} \big(B_{r_Q}(d_Q^g b_Q) - B_{r_Q}(d_Q^g b_Q - v_Q) + B_{r_Q}(-v_Q)\big).$$

We apply (3.3) to No 14 and 15. If f is of No 14 or 15, then by Lemma 3.1 and Theorem 3.2, the covering α_Y of degree g constructed in (3.1) is totally ramified at every fictitious singularity Q in I of Y. Hence $d_Q^g/(r_Q/g)$ is co-prime to g. In No 14, I consists of Q with $(r_Q, v_Q) = (r, 2)$, and $d_Q^g \geq 2$ by $r/g \geq 2$ in Theorem 3.2. In No 15, I consists of Q_1 with $(r_1, 1)$ and Q_2 with $(r_2, 1)$, and $d_{Q_1}^g, d_{Q_2}^g \geq 1$. Note that $r_1 = r_2$ and $d_1 = d_2$ in No 15'a. When f is of No 14 or 15, the equality (3.3) becomes

$$1 = \sum_{Q \in I} \left(\frac{\overline{d_Q^g b_Q}(r_Q - \overline{d_Q^g b_Q})}{2r_Q} \right.$$
$$\left. - \frac{(\overline{d_Q^g b_Q} - v_Q)(r_Q - \overline{d_Q^g b_Q} + v_Q)}{2r_Q} + \frac{v_Q(r_Q - v_Q)}{2r_Q} \right)$$
$$= \sum_{Q \in I} \frac{v_Q(r_Q - \overline{d_Q^g b_Q})}{r_Q}.$$

The lemma is deduced from this equality. $\hspace{2cm}$ q.e.d.

The idea of the proof of Lemma 3.5 is to examine the variation of the value of $B_{i,j} - B_{i,j-1}$ by the replacement of (i, j) with $(i + n', j - a')$. We shall pay attention to $A_{i,j} - A_{i,j-1}$ instead. This value encodes more information, although it is more difficult to handle. By (2.3) we obtain that

$$d(0,0) - d(n_p, -a_p) = -A_{0,-1} - (A_{n_p,-a_p} - A_{n_p,-a_p-1}).$$

If $p \geq 2$ then

$$(3.4) \qquad 1 = \sum_{Q \in I} \big(-A_Q(-e_Q) - A_Q(d_Q^p) + A_Q(d_Q^p - e_Q)\big).$$

Lemma 3.6. *a is co-prime to n if f is of No 4, 10 or 12.*

Proof. We first remark that in the case No 10 in Table 6, $e_{Q_1} = e_{Q_2} = 1$ and thus $d_1 = d_2 = \overline{a' - n'}$, whence (d_1, d_2, d_3) has to be $(1, 1, 0)$.

Suppose that a and n are not co-prime. Then they have a common divisor $p = 2$. Starting with Table 6 and Lemma 3.5(i), we compute the term $c_i := -A_{Q_i}(-e_{Q_i}) - A_{Q_i}(d_i) + A_{Q_i}(d_i - e_{Q_i})$ which appears in the right-hand side of (3.4), where $d_i := d_{Q_i}^2$.

 (i) In No 4, $(c_1, c_2) = (1/2, 0)$.
 (ii) In No 10, $(c_1, c_2, c_3) = (1/4, 1/4, 0)$.
 (iii) In No 12, $(c_1, c_2) = (1/4, 0)$. $c_3 = 0$ if $b_{Q_3} = 1$ and $c_3 = 1/2$ if $b_{Q_3} = 3$.

Hence (3.4) never holds, which is a contradiction. $\hspace{2cm}$ q.e.d.

We prove the co-primeness of a and n when f is of exceptional type except for a few cases to be investigated in Section 5, where the values of a and n are small.

Theorem 3.7. *If f is of exceptional type, then a is co-prime to n except for the following possibilities.*

   (i) *f is of No 8 and $(a, n) = (2, 2)$.*
   (ii) *f is of No 14 and $(a, n) = (2, 2)$ or $(4, 2)$.*
   (iii) *f is of No $15'$a and $(a, n) = (2, 2)$ or $(4, 2)$.*

Proof. By Corollary 3.4 and Lemma 3.6, we have only to treat the case where f is of No 14 or $15'$a. In this case $g = 1$ or 2 by Lemma 3.5(i). Suppose that $g = 2$. $a = 2a'$, $n = 2n'$, and a' is co-prime to n'. We have the following by Table 5.

   (i) If f is of No 14 then $R^*E^3 = 2n'/a'$ and $(a/n)^2 RE^3 = 4a'/n'$.
   (ii) If f is of No $15'$a then $R^*E^3 = 2n'/a'$ and $(a/n)^2 RE^3 = 2a'/n'$.

Hence $(a, n) = (2, 2), (2, 4), (2, 8)$ or $(4, 2)$ in No 14 and $(a, n) = (2, 2), (2, 4)$ or $(4, 2)$ in No $15'$a by Lemma 2.4. In order to exclude the case $n \neq 2$, we compute $d(2, 0) - d(1, 0)$ by (2.6) in terms of r_Q, b_Q for $Q \in I$. This value is, modulo $\mathbb{Z}$, equal to

$$B_{r_Q}(b_Q) - B_{r_Q}(b_Q - 2) \equiv \frac{2 - 2b_Q}{r_Q} \quad \text{for No 14, } (a, n) = (2, 4),$$

$$-\frac{1}{r_Q} + B_{r_Q}(b_Q) - B_{r_Q}(b_Q - 2) \equiv \frac{1 - 2b_Q}{r_Q} \quad \text{for No 14, } (a, n) = (2, 8),$$

$$2B_{r_Q}(b_Q) - 2B_{r_Q}(b_Q - 1) \equiv \frac{1 - 2b_Q}{r_Q} \quad \text{for No } 15'\text{a, } (a, n) = (2, 4).$$

Note that r_Q is even by Theorem 3.2. The value above becomes an integer only if f is of No 14 with $(a, n) = (2, 4)$ and $b_Q = 1$ or $r_Q/2 + 1$. In this case $e_Q = r_Q/2 + 2$ by Lemma 3.5(i), and thus $4 \mid r_Q$. This contradicts that $v_Q = \overline{e_Q b_Q} = 2$. q.e.d.

   To treat the cases left in Theorem 3.7, we are forced to analyse f geometrically by making use of Corollary 4.2. This is the reason why we postpone completing the proof of Theorem 1.3. We here collect the numerical data to be used.

Lemma 3.8. *The cases left in Theorem 3.7 have the following data.*

   (i) *f is of No 8. I consists of $Q = (6, 2)$ and $Q' = (2, 1)$. $(a, n) = (2, 2)$. $e_Q = 4$ and $b_Q = 5$. $d(0, -1) = 0$ and $d(0, -2) = 1$.*
   (ii) *f is of No 14. I consists of $Q = (2r', 2)$. $n = 2$. (a, b_Q) is either $(2, r' + 2)$ with $r' \geq 3$ or $(4, r' + 4)$ with $r' \geq 5$. $d(0, -1) = 1$ and $d(-1, 1) = 0$.*
   (iii) *f is of No $15'$a. I consists of two fictitious singularities with data $(2r', 1)$ from one singularity Q of Y. $n = 2$. (a, b_Q) is either $(2, r' + 1)$ with $r' \geq 2$ or $(4, r' + 2)$ with $r' \geq 3$. $d(0, -1) = 1$ and $d(-1, 1) = 0$.*

Proof. All the data are deduced from Theorem 2.3, (2.6), (2.9), Table 6 and Lemmata 3.3, 3.5(i). Lemma 3.3 is used to estimate r'. q.e.d.

4. General elephant theorem

In this section we prove the general elephant Theorem 1.4 in strong form by either of two different ways according to the value of the discrepancy a/n.

The theorem in the strongest form is easily deduced from its local version when the discrepancy a/n is small. In this case we can obtain a general elephant of Y by taking the birational transform of a general elephant of X.

Theorem 4.1. *If $d(-1, i) = 0$ for every $1 \le i < a/n$, then the birational transform on Y of a general elephant of X is a general elephant of Y and has at worst Du Val singularities. This is the case in particular when $a/n \le 1$.*

Proof. Let S_X be a general elephant of X and S its birational transform on Y. S_X has at worst a Du Val singularity, or equivalently, a canonical singularity at P. We set $f^* S_X = S + (a/n + m)E$ by an integer $m > -a/n$. The assumption on $d(-1, i)$ is by (2.8) equivalent to $m \ge 0$. On the other hand, by the adjunction formula, ω_S is in codimension one equal to $f^* \omega_{S_X} \otimes \mathcal{O}_Y(-mE)$, whence $m \le 0$ since S_X is canonical. Thus $m = 0$ and $S \to S_X$ is a crepant morphism. Now the theorem follows obviously.   q.e.d.

Theorem 4.1 is applied to the case where f is of exceptional type by Corollary 2.5, Theorem 3.7 and Lemma 3.8.

Corollary 4.2. *If f is of exceptional type, then the assumption in Theorem 4.1 holds and consequently the general elephant theorem holds for f.*

Therefore, with regard to the general elephant theorem, it suffices to treat the case where f is of general type with discrepancy $a/n > 1$. We keep the following notation when f is of general type.

   (i) If f is of No 15″, then I consists of $Q_1 = (r_1, 1)$ and $Q_2 = (r_2, 1)$. Set $b_1 := b_{Q_1}$ and $b_2 := b_{Q_2}$.
   (ii) If f is of No 16, then I consists of $Q = (r, 1)$. Set $b := b_Q$. We set $r_1 := 1$, $r_2 := r$ and $b_1 := 0$, $b_2 := b$ to treat both the cases simultaneously.

We have $(a/n)E^3 = 1/r_1 + 1/r_2$ by Theorem 2.3. We let S be a general elephant of Y and S_X its birational transform on X. The general elephant theorem guarantees that S_X has at worst a Du Val singularity at P and the induced morphism $f_S \colon S \to S_X$ is crepant. In our case we prove the theorem with the description of this partial resolution.

Theorem 4.3. *If f is of general type, then a general elephant S of Y has at worst Du Val singularities. Moreover, except for the case $a = 1$, the dual graph (see Section 5 for notation) for the partial resolution $f_S \colon S \to S_X$ is one of the following.*

   (i)
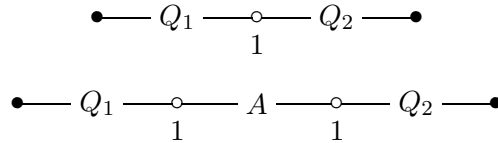
*The attached number at ◦ is the coefficient in the fundamental cycle.
In No 15″, Q_1, Q_2 correspond to the points Q_1, Q_2 in I, and in No
16, Q_1, Q_2 correspond to a smooth point and Q in I. S has Du
Val singularities of types A_{r_1-1}, A_{r_2-1}, A respectively at Q_1, Q_2, A
(possibly smooth at A), and they are all the singularities of S. In
particular, S_X has a Du Val singularity of type A and consequently
P is of type cA/n.*

(ii)
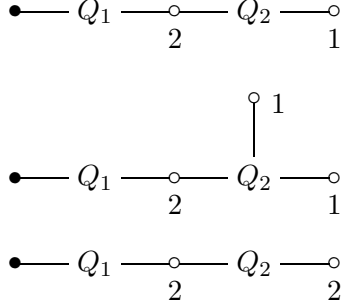
In this case f is of No 15″ with $n = 1$ or 2, and either
 (a) $r_2 = r_1 + n$ with $a \mid 2r_1 + n$ and a is odd; or
 (b) $r_2 = r_1 + 2n$ with $a \mid r_1 + n$.
*In each case $a < r_1 < r_2$, $n \mid a + r_i$ and $b_i \equiv (a + r_i)/n$ modulo r_i
for $i = 1, 2$. Q_1, Q_2 correspond to the points Q_1, Q_2 in I. S has
Du Val singularities of types A_{r_1-1}, A_{r_2-1} respectively at Q_1, Q_2,
and they are all the singularities of S unless S has the last graph.
In the case of the last graph, S may have further singularities on
the exceptional curve not passing through Q_1. If $n = 2$ then the
assumption in Theorem 4.1 holds.*

We let f be a divisorial contraction of general type. We start with the
computation of $d(0, -1)$ and $d(-1, 0)$. By (2.6), we obtain that

$$d(1, -1) - d(0, -1) = -\frac{3}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \sum_{i=1,2}\left(\frac{r_i - 1}{2r_i} - \frac{\overline{2}(r_i - \overline{2})}{2r_i}\right),$$

$$d(0, 0) - d(-1, 0) = -\left(\frac{a}{n} + \frac{1}{2}\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$
$$+ \sum_{i=1,2}\left(\frac{b_i(r_i - b_i)}{2r_i} - \frac{(b_i + 1)(r_i - b_i - 1)}{2r_i}\right).$$

We note that $d(0, 0) = 1$ by (2.9). Hence

(4.1) $$d(0, -1) = d(1, -1) + \begin{cases} 1 & \text{for No 15,} \\ 2 & \text{for No 16,} \end{cases}$$

(4.2) $$d(-1, 0) = 2 + \sum_{i=1,2}\left(\frac{a}{n} - b_i\right)\frac{1}{r_i} \geq 1.$$

By (2.8) and (4.1), we have $d(1, -1) \geq 0$ and a general hyperplane section
H_X on the germ $P \in X$ has multiplicity 1 along E, that is, $f^*H_X = H + E$,
where H is the birational transform on Y of H_X. By (2.8) and (4.2), there

exists an effective divisor S_X on X which is linearly equivalent to $-K_X$ and has multiplicity a/n along E, that is, $f^*S_X = S + (a/n)E$, where S is the birational transform on Y of S_X. If we take S_X generally then the surface S is a general elephant of Y. Note that S is not necessarily a prime divisor. S passes through all the singularities in I_0, whereas H passes through all those in I. By Corollary 2.7, the curves $H \cap E$ and $S \cap E$ have no embedded points, and their reduced loci are trees of $\mathbb{P}^1$. We reduce the case (i) of Theorem 4.3 to the following more approachable statement.

Lemma 4.4. *Theorem* 4.3(i) *follows from one of the following.*

(i) $H \cap E$ *has a* (*possibly reducible*) *component which passes through all the points in I and which intersects S properly at two distinct points.*

(ii) $S \cap E$ *has a* (*possibly reducible*) *component which intersects H properly at two distinct points.*

Proof. The lemma is proved in the same fashion on either assumption. Here we demonstrate it by assuming (i). We let C denote the component of $H \cap E$ in the assumption. Since $(S \cdot C) \le (S \cdot H \cdot E) = (a/n)E^3 = 1/r_1 + 1/r_2$, the two points on $S \cap C$ are Q_1 and Q_2 in No 15'', and a smooth point Q_1 and $Q_2 := Q$ in No 16. Moreover $C = H \cap E$ and $(S \cdot H \cdot E)_{Q_1} = 1/r_1$, $(S \cdot H \cdot E)_{Q_2} = 1/r_2$. Hence S is a Du Val singularity of type A_{r_i-1} at Q_i for $i = 1, 2$, and $S \cap E$ is irreducible and reduced at each Q_i. By the f-ampleness of H, every irreducible component of $S \cap E$ passes through either Q_1 or Q_2. Hence $S \cap E$ is isomorphic to $\mathbb{P}^1$ or the tree of two $\mathbb{P}^1$ with a node Q' as pictured below.
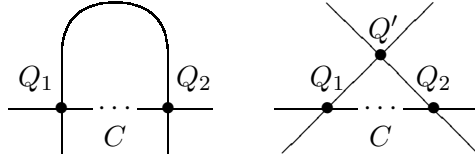
Because E is Cartier outside Q_1 and Q_2, S is smooth outside Q_1, Q_2 and the singularities of $S \cap E$. The only possible singularity of $S \cap E$ is a node Q', at which S has embedding dimension 2 or 3. Suppose that S is singular at Q'. In order that $S \cap E$ has a node Q', the surface S in the tangent space $T_{Q'}S = \mathbb{C}^3$ has to be given by the equation $x_1x_2 + x_3 \cdot (\text{something}) = 0$ for suitable local coordinates x_1, x_2, x_3 at Q', where E is given by $x_3 = 0$. Hence S has a Du Val singularity of type A at Q', and the lemma is deduced.   q.e.d.

The rest of this section is devoted to the proof of Theorem 4.3 on the assumption that f has discrepancy $a/n > 1$. From now on, we keep the assumption that f is of general type with discrepancy $a/n > 1$. The theorem is proved when $n = 1$ by [14, Section 5] and Erratum attached at the end of this paper. We here assume that $n \ge 2$. Because $(H \cdot H \cdot E) = E^3 = (n/a)(1/r_1 + 1/r_2) < 1/r_1 + 1/r_2$, there exists an irreducible reduced component of $H \cap E$ which passes through all the singularities in I, that is, Q_1 and Q_2 in No 15'' and Q in No 16. We let $C \cong \mathbb{P}^1$ be such an irreducible reduced component of $H \cap E$. $Q_1, Q_2 \in C$ in No 15'' and $Q \in C$ in No 16. We take a normal form explained at the end of Section 2 at each singularity

in I. The curve C is determined uniquely in No $15''$ since $(H \cap E)_{\mathrm{red}}$ is a tree of $\mathbb{P}^1$, but in No 16 every irreducible component of $H \cap E$ passes through Q. When f is of No 16, we choose C wisely as in the following lemma.

Lemma 4.5. *If f is of No 16, then there exists an effective divisor D_X on X whose multiplicity along E is $1/n$, that is, $f^*D_X = D + (1/n)E$, where D is the birational transform on Y of D_X. Moreover, by taking a suitable H, we can choose C so that D intersects C properly and so that there exists no hidden non-Gorenstein point on C.*

Proof. a is co-prime to n by Theorem 3.2. Thus there exist integers s, t such that $as + nt = -1$. The existence of D_X follows if $d(s,t) \neq 0$ by (2.8). By (2.6), we obtain that

$$d(s+1,t) - d(s,t) = -\left(\frac{1}{n} + \frac{1}{2}\right)\left(1 + \frac{1}{r}\right)$$
$$+ \left(\frac{(u+1)(r-u-1)}{2r} - \frac{u(r-u)}{2r}\right),$$

where $u := \overline{sb + t - 1}$. Since $d(s+1,t) = 0$ by (2.9), we have

$$d(s,t) = \frac{1}{n}\left(1 + \frac{1}{r}\right) + \frac{u+1}{r} > 0,$$

whence the existence of D_X is deduced.

We have the surjective map $f_*\mathcal{O}_Y(H) \twoheadrightarrow H^0(\mathcal{Q}_{0,-1})$ by Lemma 2.6 and the estimate $h^0(\mathcal{Q}_{0,-1}) = d(0,-1) = 2$ by (2.7) and (4.1). Hence the restriction on E of the divisor H moves in its linear system, although we need to mind the possibility that E is non-normal. Thus we can take H so that $H \cap E$ has an irreducible reduced component C which intersects D properly. Suppose that there exists a hidden non-Gorenstein point Q' on C. Then the index r' at Q' is a divisor of n, whence $D \sim sK_Y + tE$ is not Cartier at Q' and consequently D passes through Q'. However, this contradicts that $((D + H) \cap E)_{\mathrm{red}}$ is a tree of $\mathbb{P}^1$ in Corollary 2.7 by $a/n > 1$. Therefore Q is the only non-Gorenstein point which C passes through.   q.e.d.

We consider $D_{i,j}$ in (2.1) and write $s_C(i,j)$ for the integer such that

$$(\mathcal{O}_Y(D_{i,j}) \otimes \mathcal{O}_C)/(\text{torsion}) \cong \mathcal{O}_{\mathbb{P}^1}(s_C(i,j)).$$

Note that $s_C(i,j)$ is well-defined for any C on Y isomorphic to $\mathbb{P}^1$. We give a condition for Theorem 4.3(i) to hold in terms of the value of $s_C(-1,0)$.

Lemma 4.6. *Theorem 4.3(i) holds if and only if $s_C(-1,0) = 0$ in No $15''$ or $s_C(-1,0) = 1$ in No 16. In this case, for any point Q in I (Q_1 or Q_2 in No $15''$ and Q in No 16) we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, b_Q)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that*

$$Q^\sharp \in C^\sharp \subset Y^\sharp \cong o \in (x_3\text{-axis}) \subset \mathbb{C}^3_{x_1 x_2 x_3}.$$

Proof. Let R_0 denote the global Gorenstein index of Y. Compute the image of the map

$$\mathcal{O}_Y(-K_Y)^{\otimes R_0} \otimes \mathcal{O}_Y(R_0 K_Y) \otimes \mathcal{O}_C \to \mathcal{O}_C.$$

By (2.14), its image at any $Q \in I$ is $\mathcal{O}_C(-R_0 w_Q^C(b_Q)/r_Q \cdot Q)$. Hence the image is contained in the ideal sheaf isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-R_0(w_{Q_1}^C(b_1)/r_1 + w_{Q_2}^C(b_2)/r_2))$ in No 15″ and $\mathcal{O}_{\mathbb{P}^1}(-R_0 w_Q^C(b)/r)$ in No 16. On the other hand, this image equals that of

$$\left((\mathcal{O}_Y(-K_Y)^{\otimes R_0} \otimes \mathcal{O}_C)/(\text{torsion})\right) \otimes \left(\mathcal{O}_Y(R_0 K_Y) \otimes \mathcal{O}_C\right) \to \mathcal{O}_C,$$

whence

$$R_0 s_C(-1, 0) + R_0(K_Y \cdot C) \leq \begin{cases} -R_0 \left(\dfrac{w_{Q_1}^C(b_1)}{r_1} + \dfrac{w_{Q_2}^C(b_2)}{r_2} \right) & \text{for No 15″}, \\ -R_0 \cdot \dfrac{w_Q^C(b)}{r} & \text{for No 16}. \end{cases}$$

Note that by Lemma 4.5, in fact the equality holds if f is of No 16. If Theorem 4.3(i) holds, then $C = H \cap E$, $w_Q^C(b_Q) = 1$ for any $Q \in I$ and the above inequality becomes an equality by the proof of Lemma 4.4(i), whence the condition on $s_C(-1, 0)$ holds. On the other hand if the assumption on $s_C(-1, 0)$ holds, then the above inequality becomes

$$(-K_Y \cdot C) \geq \begin{cases} \dfrac{w_{Q_1}^C(b_1)}{r_1} + \dfrac{w_{Q_2}^C(b_2)}{r_2} & \text{for No 15″}, \\ 1 + \dfrac{w_Q^C(b)}{r} & \text{for No 16}. \end{cases}$$

We have $(-K_Y \cdot C) \leq (-K_Y \cdot H \cdot E) = 1/r_1 + 1/r_2$. Hence we obtain that $C = H \cap E$, $w_Q^C(b_Q) = 1$ for every $Q \in I$, and there exists no hidden non-Gorenstein point on C. The desired coordinates at Q are constructed by $w_Q^C(b_Q) = 1$. By applying Lemma 2.6 to $(i, j; i', j') = (0, -1; -1, 0)$, $L_{0,-1} = H$ and $D_{-1,0} = -K_Y$, we obtain the surjective map

$$f_* \mathcal{O}_Y(-K_Y) \twoheadrightarrow H^0(\mathcal{R}_{0,-1}^{-1,0}) \twoheadrightarrow H^0((\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_C)/(\text{torsion}))$$
$$= \begin{cases} H^0(\mathcal{O}_{\mathbb{P}^1}) & \text{for No 15″}, \\ H^0(\mathcal{O}_{\mathbb{P}^1}(1)) & \text{for No 16}. \end{cases}$$

This surjective map and the equality $(S \cdot C) = 1/r_1 + 1/r_2$ imply that S intersects C properly at Q_1 and Q_2 in No 15″, and at Q and another point in No 16. Hence Theorem 4.3(i) holds by Lemma 4.4(i). q.e.d.

Unlike the case where X is Gorenstein, C might pass through hidden non-Gorenstein points in No 15″. In spite of this, the proof of Lemma 4.6 suggests that there exists no hidden non-Gorenstein point on C even if f is of No 15″. We desire this property especially in using the formula (2.14). We prove that this is the case after a lemma on the values of $s_C(i, j)$ which is used frequently.

Lemma 4.7. (i) *If f is of No 15″, then*
(a) *$s_C(i, j) < 0$ and $s_C(-i, -j) \geq -1$ if $ia/n + j > 0$.*
(b) *$s_C(i, j) = -1$ and $s_C(-i, -j) = 0$ or -1 if $0 < ia/n + j < a/n$,*
    where $s_C(-i, -j) = 0$ can occur only if $ib_1 + j \equiv 0$ modulo r_1.
(c) *$s_C(i, j) = -1$ if $ia/n + j = 0$ with $(i, j) \notin \mathbb{Z}(n, -a)$.*
(d) *$(s_C(1, 0), s_C(-1, 0)) = (-2, 0)$, $(-2, -1)$ or $(-1, -1)$.*
(e) *$s_C(1, 1) = -1$.*

(ii) *If f is of No 16, then*
    (a) $s_C(i,j) < 0$ *and* $s_C(-i,-j) \geq 0$ *if* $ia/n + j > 0$.
    (b) $s_C(i,j) = -1$ *and* $s_C(-i,-j) = 0$ *if* $0 < ia/n + j < a/n$.
    (c) $(s_C(1,0), s_C(-1,0)) = (-2,1)$, $(-2,0)$ *or* $(-1,0)$.

Proof. Let R_0 be the global Gorenstein index of Y. For (i), we have $s_C(i,j) < 0$ for $ia/n + j > 0$ by the existence of the natural map $\mathcal{O}_Y(iK_Y + jE)^{\otimes R_0} \otimes \mathcal{O}_C \to \mathcal{O}_Y(R_0(iK_Y + jE)) \otimes \mathcal{O}_C$. By the surjective map $H^1(\mathcal{R}_{0,-1}^{i,j}) \twoheadrightarrow H^1(\mathcal{O}_{\mathbb{P}^1}(s_C(i,j)))$ and Lemma 2.6, we obtain that $s_C(i,j) \geq -1$ if $ia/n+j < a/n$ and that $s_C(1,0) = -1$ or -2. We have (a)-(d) by considering that the map $\mathcal{O}_Y(iK_Y + jE) \otimes \mathcal{O}_Y(-iK_Y - jE) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is not surjective at Q unless $iK_Y + jE$ is Cartier at Q. Note that $iK_Y + jE$ is not Cartier at Q_2 for $0 < ia/n + j \leq a/n$ by $a < r_2$ in Lemma 3.3. By Lemma 2.6, (2.9) and (4.1), we have $h^1(\mathcal{Q}_{1,1}) = 0$, $h^2(\mathcal{Q}_{1,1}) = d(0,0) = 1$ and $h^2(\mathcal{Q}_{1,2}) = d(0,-1) = 1$. Hence $h^1(\mathcal{R}_{0,-1}^{1,1}) = 0$ by (2.11) and (e) follows.

    (ii) is proved in the same way once we obtain the isomorphism $(\mathcal{O}_Y(D) \otimes \mathcal{O}_C)/(\text{torsion}) \cong \mathcal{O}_{\mathbb{P}^1}$ for $D = D_{s,t}$ consructed in Lemma 4.5. Since $(D \cdot C) = (1/n)(H \cdot C) = (1/a)(1 + 1/r) < 1$, we have $(D \cdot C)_Q = (D \cdot C)$. Let $D^\sharp$ denote the pre-image of D on the index-one cover $Q^\sharp \in Y^\sharp$. Then the restriction of the defining function of $D^\sharp$, whose weight is $-(sb+t)$, to $C^\dagger$ has order $(D \cdot C)_Q$ with respect to t, whence $w_Q^C(-sb - t) = r(D \cdot C)_Q = r(D \cdot C)$. From the map $\mathcal{O}_Y(D)^{\otimes r} \otimes \mathcal{O}_Y(-rD) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we obtain that $rs_C(s,t) - r(D \cdot C) = -w_Q^C(-sb - t)$. Hence $s_C(s,t) = 0$.   q.e.d.

Lemma 4.8. *There exists no hidden non-Gorenstein point on C.*

Proof. By Lemma 4.5, we have only to consider the case No $15''$. We shall prove the lemma for No $15''$ by using the maps

$$(4.3) \qquad \mathcal{O}_Y(iK_Y + jE) \otimes \mathcal{O}_Y(-iK_Y - jE) \otimes \mathcal{O}_C \to \mathcal{O}_C$$

for various i, j. Note that the map (4.3) is not surjective at every non-Gorenstein point on C at which $iK_Y + jE$ is not Cartier. In particular, the number of those points is at most $-s_C(i,j) - s_C(-i,-j)$. Suppose that there exists a hidden non-Gorenstein point Q' on C. We apply this principle with Lemma 4.7(i).

    For each $i = 1, 2$, at least one of the maps (4.3) for $(i,j) = (1,-1)$ and $(1,-2)$ is not surjective at Q_i, and both of the maps are not surjective at Q'. Since $-s_C(i,j) - s_C(-i,-j) \leq 2$ for these (i,j) by Lemma 4.7(i), Q' is the only hidden non-Gorenstein point on C, and that $b_A = 1$, $b_B = 2$ for $(A,B) = (1,2)$ or $(2,1)$. If $a/n \geq 2$, then by the map (4.3) for $(i,j) = (1,-3)$, the divisor $K_Y - 3E$ has to be Cartier at Q_1 or Q_2, whence $(A,B) = (1,2)$ and $r_1 = 2$. In particular, r_1 is co-prime to r_2 and consequently a is co-prime to n by Theorem 3.2. Thus $a/n > 2$ and the map (4.3) for $(i,j) = (1,-4)$ is not surjective at Q_1, Q_2 and Q'. This contradicts that $-s_C(1,-4) - s_C(-1,4) = 2$ in Lemma 4.7(i). Therefore $1 < a/n < 2$. The map (4.3) for $(i,j) = (2,-3)$ is not surjective at Q_1, Q_2 by $b_A = 1$, $b_B = 2$. Hence this map has to be surjective at Q', or equivalently the index $r_{Q'}$ of Y at Q' is 2. However also the map (4.3) for $(i,j) = (3,-4)$ is not surjective at Q_1, Q_2, whence $r_{Q'}$ must be 3. This is a contradiction.   q.e.d.

We proceed to the delicate analysis of the embedding of C by using the concept of normal form in Section 2. We need to keep Lemma 4.8 in mind. First we prove that the embedding dimension of $C^\sharp$ is at most two. This dimension equals the number of generators of the semi-group G_Q^C in (2.13).

Lemma 4.9. *Consider any point Q in I (Q_1 or Q_2 in No 15″ and Q in No 16), and take a normal form of $Q \in C \subset Y$. Then the semi-group G_Q^C is generated either by $(a_1, 1)$ and (a_3, b_Q) or by $(a_2, -1)$ and (a_3, b_Q). In particular, we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, b_Q)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{c_Q/r_Q}, 0, t^{\overline{b_Q c_Q}/r_Q})$ or $(0, t^{c_Q/r_Q}, t^{1-\overline{b_Q c_Q}/r_Q})$ for some positive integer $c_Q < r_Q$.*

Proof. Because C is smooth at Q, there exists an invariant monomial in x_1, x_2, x_3 whose restriction to C is t, or equivalently $(r_Q, 0) \in G_Q^C$. Hence one of the following holds.

(i) One of a_1, a_2, a_3 is 1.
(ii) $(r_Q, 0)$ is generated either by $(a_1, 1)$ and (a_3, b_Q) or by $(a_2, -1)$ and (a_3, b_Q) in $\mathbb{Z} \times \mathbb{Z}/(r_Q)$.
(iii) $r_Q = a_1 + a_2$.

The lemma follows directly if (i) or (ii) holds. From now on we suppose that only the statement (iii) holds. In particular $r_Q \geq 5$. On this assumption, the inequality $w_Q^C(k) + w_Q^C(-k) \geq 2r_Q$ holds unless $k = 0$ or ± 1.

If a is not co-prime to n, then f is of No 15″ by Theorem 3.2 and we have a numerically f-trivial divisor $D_{n', -a'}$, which is not Cartier at Q_1 and Q_2 by Lemma 3.5(ii). By the map $\mathcal{O}_Y(-D_{n', -a'}) \otimes \mathcal{O}_Y(D_{n', -a'}) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have

$$s_C(-n', a') + s_C(n', -a') = -\sum_{i=1,2}\left(\frac{w_{Q_i}^C(n'b_i - a')}{r_i} + \frac{w_{Q_i}^C(a' - n'b_i)}{r_i}\right).$$

Since $s_C(n', -a') = s_C(-n', a') = -1$ by Lemma 4.7(i), we have $w_Q^C(n'b_Q - a') + w_Q^C(a' - n'b_Q) = r_Q$. Hence (i) or (ii) above must occur.

Now we may assume that a is co-prime to n. We take integers s, t so that $as + nt = -1$ and consider the divisor $D_{s,t} \equiv -(1/n)E$. Then for each $Q \in I$, r_Q is the smallest positive integer such that $r_Q D_{s,t}$ is Cartier. Consider the map $\mathcal{O}_Y(-iD_{s,t}) \otimes \mathcal{O}_Y(iD_{s,t}) \otimes \mathcal{O}_C \to \mathcal{O}_C$. We have

$$s_C(-is, -it) + s_C(is, it) = -\sum_{Q \in I}\left(\frac{w_Q^C(i(sb_Q + t))}{r_Q} + \frac{w_Q^C(-i(sb_Q + t))}{r_Q}\right).$$

By this equality and Lemma 4.7,

$$(4.4) \qquad w_Q^C(i(sb_Q + t)) + w_Q^C(-i(sb_Q + t)) = r_Q$$

for $0 < i < a$ except that f is of No 15″ with $r_1 \mid i$. On the other hand, $w_Q^C(k) + w_Q^C(-k) \geq 2r_Q$ unless $k = 0$ or ± 1 by the assumption. First by (4.4) for $i = 1$, we have $sb_Q + t \equiv \pm 1$ modulo r_Q. Then $2(sb_Q + t), 3(sb_Q + t) \not\equiv 0, \pm 1$ modulo r_Q by $r_Q \geq 5$. Thus by (4.4) for $i = 2$, f is of No 15″, $Q = Q_2$ and $r_1 = 2$. Hence by (4.4) for $i = 3$ and $a/n > 1$, we have $(a, n) = (3, 2)$. This contradicts that $n \equiv ae_{Q_1}$ modulo r_1.   q.e.d.

The precise generators of G_Q^C are written as follows.

Lemma 4.10.   (i) *If f is of No 15″, then we may choose the coordinates so that the restriction to $C^\dagger$ at one of Q_1 and Q_2 is of form $(t^{c_i/r_i}, 0, t^{\overline{b_i c_i}/r_i})$ and that at the other is of form $(0, t^{c_i/r_i}, t^{1-\overline{b_i c_i}/r_i})$.*
   (ii) *If f is of No 16, then we may choose the coordinates so that the restriction to $C^\dagger$ at Q is of form $(t^{c/r}, 0, t^{\overline{bc}/r})$.*

Proof. We note that the lemma holds on the assumption in Lemma 4.6. Starting with Lemma 4.9, first we consider the case No 15″. If both are of form $(t^{c_i/r_i}, 0, t^{\overline{b_i c_i}/r_i})$, then the image of the map $\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_Y(-E) \otimes \mathcal{O}_Y(K_Y + E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is $\mathcal{O}_C(-Q_1 - Q_2)$, whence $s_C(-1,0) + s_C(0,-1) + s_C(1,1) = -2$. Thus $s_C(-1,0) = 0$ by Lemma 4.7(i). If both are of form $(0, t^{c_i/r_i}, t^{1-\overline{b_i c_i}/r_i})$, then the image of the map $\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_Y(E) \otimes \mathcal{O}_Y(K_Y - E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is $\mathcal{O}_C(-Q_1 - Q_2)$, whence $s_C(-1,0) + s_C(0,1) + s_C(1,-1) = -2$. Thus $s_C(-1,0) = 0$ by Lemma 4.7(i). If f is of No 16 and the restriction is of form $(0, t^{c/r}, t^{1-\overline{bc}/r})$, then the image of the map $\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_Y(E) \otimes \mathcal{O}_Y(K_Y - E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is $\mathcal{O}_C(-Q)$, whence $s_C(-1,0) + s_C(0,1) + s_C(1,-1) = -1$. Thus $s_C(-1,0) = 1$ by Lemma 4.7(ii).   q.e.d.

It is possible now to derive Theorem 4.3 when f is of No 16 with discrepancy $a/n > 1$. The divisor D constructed in Lemma 4.5 plays a crucial role in the proof.

Theorem 4.11. *If f is of No 16 with discrepancy $a/n > 1$, then Theorem 4.3 holds.*

Proof. By virtue of Lemma 4.6, it suffices to prove that $s_C(-1,0) = 1$. First we shall deduce that $\overline{bc} < c$ in Lemma 4.10(ii) by using the effective divisor $D \sim sK_Y + tE \equiv -(1/n)E$ constructed in Lemma 4.5. From the map $\mathcal{O}_Y(D)^{\otimes r} \otimes \mathcal{O}_Y(-rD) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(s,t) - r(D \cdot C) = -w_Q^C(-sb - t)$, whence

$$\frac{1}{n}(H \cdot C) = (D \cdot C) = \frac{w_Q^C(-sb - t)}{r}$$

by $s_C(s,t) = 0$ in Lemma 4.7(ii). On the other hand, from the map $\mathcal{O}_Y(H)^{\otimes r} \otimes \mathcal{O}_Y(-rH) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(0,-1) - r(H \cdot C) = -w_Q^C(1) = -c$, whence

$$(4.5) \qquad\qquad (H \cdot C) = \frac{c}{r}$$

by $s_C(0,-1) = 0$ in Lemma 4.7(ii). Hence $w_Q^C(-sb - t) = c/n < c$. Because G_Q^C in (2.13) is generated by $(c, 1)$ and $(\overline{bc}, b)$, this inequality implies that $\overline{bc} < c$.

From the map $\mathcal{O}_Y(-K_Y)^{\otimes r} \otimes \mathcal{O}_Y(rK_Y) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $rs_C(-1,0) + r(K_Y \cdot C) = -w_Q^C(b) = -\overline{bc}$, whence

$$(4.6) \qquad\qquad \frac{a}{n}(H \cdot C) = (-K_Y \cdot C) = \frac{\overline{bc}}{r} + s_C(-1,0).$$

By (4.5), (4.6) and the inequality $\overline{bc} < c$, we have $s_C(-1,0) > 0$. Therefore the theorem follows from Lemma 4.7(ii).   q.e.d.

By virtue of Lemma 4.6, we assume that f is of No 15″ with $s_C(-1,0) \neq 0$ from now on. The goal of the rest of this section is to derive Theorem 4.3(ii) on this assumption. The embedding dimension of $C^\sharp$ is one in the situation of Lemma 4.6, but in fact this is always the case.

Lemma 4.12. *Suppose that f is of No 15″ with $s_C(-1,0) \neq 0$. Then for each $i = 1, 2$, we may choose semi-invariant local coordinates x_{i1}, x_{i2}, x_{i3} with weights $\mathrm{wt}(x_{i1}, x_{i2}, x_{i3}) = (1, -1, b_i)$ of the index-one cover $Q_i^\sharp \in Y^\sharp$ so that*

$$Q_i^\sharp \in C^\sharp \subset Y^\sharp \cong o \in (x_{ii}\text{-axis}) \subset \mathbb{C}^3_{x_{i1}x_{i2}x_{i3}}.$$

Moreover $C \subset S$, $3 \leq r_1 < r_2$ and $(H \cdot C) = 1/r_1 - 1/r_2$, $(-K_Y \cdot C) = b_1/r_1 - b_2/r_2$.

Proof. We note that $s_C(-1,0) = -1$ by Lemma 4.7(i), and the assumption $r_1 \geq 3$ is imposed by the proof of Lemma 4.10(i). Moreover, according to Lemmata 4.4(i) and 4.6, if S intersects C properly then eventually $s_C(-1,0) = 0$. Hence in our case we have $C \subset S$. By Lemma 4.10(i), we can choose $(A, B) = (1, 2)$ or $(2, 1)$ so that the description at Q_A is of form $(t^{c_A/r_A}, 0, t^{\overline{b_A c_A}/r_A})$ and that at Q_B is of form $(0, t^{c_B/r_B}, t^{1-\overline{b_B c_B}/r_B})$.

Step 1. $c_A < \overline{b_A c_A}$.

Indeed, by the map $\mathcal{O}_Y(H)^{\otimes r_A r_B} \otimes \mathcal{O}_Y(-r_A r_B H) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $r_A r_B s_C(0, -1) - r_A r_B (H \cdot C) = -r_B w_{Q_A}^C(1) - r_A w_{Q_B}^C(1)$. By $w_{Q_A}^C(1) = c_A$, $w_{Q_B}^C(1) = r_B - c_B$ and $s_C(0, -1) = -1$ in Lemma 4.7(i), we obtain that

$$(4.7) \qquad\qquad 1 + (H \cdot C) = \frac{c_A}{r_A} + \frac{r_B - c_B}{r_B}.$$

On the other hand, by the map $\mathcal{O}_Y(-K_Y)^{\otimes r_A r_B} \otimes \mathcal{O}_Y(r_A r_B K_Y) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $r_A r_B s_C(-1, 0) + r_A r_B(K_Y \cdot C) = -r_B w_{Q_A}^C(b_A) - r_A w_{Q_B}^C(b_B)$. By $w_{Q_A}^C(b_A) = \overline{b_A c_A}$, $w_{Q_B}^C(b_B) = r_B - \overline{b_B c_B}$ and $s_C(-1, 0) = -1$, we obtain that

$$(4.8) \qquad\qquad 1 + \frac{a}{n}(H \cdot C) = \frac{\overline{b_A c_A}}{r_A} + \frac{r_B - \overline{b_B c_B}}{r_B}.$$

From the equalities (4.7) and (4.8), either $c_A < \overline{b_A c_A}$ or $c_B > \overline{b_B c_B}$ holds. However, $c_B > \overline{b_B c_B}$ never occurs since $(r_B, 0)$ is contained in the semi-group $G_{Q_B}^C$ in (2.13). Hence Step 1 is deduced.

Step 2. $c_A = 1$.

For Step 2, it suffices to prove the equality $b_A c_A = \overline{b_A c_A}$ by $(r_A, 0) \in G_{Q_A}^C$. Suppose that $b_A c_A > \overline{b_A c_A}$. Then $b_A \geq 3$ by $c_A < \overline{b_A c_A}$. Moreover, $(\overline{b_A c_A} - c_A, b_A - 1)$ is not contained in $G_{Q_A}^C$, and either $(\overline{b_A c_A} - 2c_A, b_A - 2)$ or $(\overline{2c_A - b_A c_A}, 2 - b_A)$ is not contained in $G_{Q_A}^C$. Therefore, for $j = 1, 2$, the image of the map $\mathcal{O}_Y(-K_Y + jE) \otimes \mathcal{O}_Y(K_Y - jE) \otimes \mathcal{O}_C \to \mathcal{O}_C$ is contained in $\mathcal{O}_C(-2Q_A)$, whence $s_C(-1, j) + s_C(1, -j) \leq -3$ unless $K_Y - jE$ is Cartier

at Q_B. It is however impossible by $a/n > 1$ and Lemma 4.7(i). Hence Step 2 follows.

Now by substituting $c_A = 1$ in (4.7) and (4.8), we have $(A, B) = (1, 2)$, $r_1 < r_2$, $(H \cdot C) = 1/r_1 - c_2/r_2$ and $(-K_Y \cdot C) = (a/n)(H \cdot C) = b_1/r_1 - \overline{b_2 c_2}/r_2$. It remains proving that $c_2 = 1$.

Step 3. $c_2 = 1$ except for the case where $b_1 = r_1 - 1$ and $d(0, -2) \geq 2$.

We have $1 - (H \cdot C) = (r_1 - 1)/r_1 + c_2/r_2$ and $1 + (a/n)(H \cdot C) = b_1/r_1 + (r_2 - \overline{b_2 c_2})/r_2$. These equalities imply that $(r_1 - 1)/r_1 + c_2/r_2 < b_1/r_1 + (r_2 - \overline{b_2 c_2})/r_2$, whence $c_2 < r_2 - \overline{b_2 c_2}$. Thus $c_2 = 1$ follows if $(r_2 - \overline{b_2 c_2} - c_2, b_2 + 1)$ is contained in $G_{Q_2}^C$. Consider the map $\mathcal{O}_Y(K_Y + E) \otimes \mathcal{O}_Y(-K_Y - E) \otimes \mathcal{O}_C \to \mathcal{O}_C$. Then $(r_2 - \overline{b_2 c_2} - c_2, b_2 + 1) \in G_{Q_2}^C$ holds if the image of this map at Q_2 is $\mathcal{O}_C(-Q_2)$. Since $s_C(1, 1) = -1$ and $s(-1, -1) \geq -1$ by Lemma 4.7(i), the image at Q_2 is $\mathcal{O}_C(-Q_2)$ unless $K_Y + E$ is Cartier at Q_1, or equivalently $b_1 = r_1 - 1$.

Similarly, the equality $c_2 = 1$ follows if $(r_2 - \overline{b_2 c_2} - 2c_2, b_2 + 2)$ and $(\overline{b_2 c_2 + 2c_2}, -b_2 - 2)$ are contained in $G_{Q_2}^C$, and this is the case if the image of the map $\mathcal{O}_Y(K_Y + 2E) \otimes \mathcal{O}_Y(-K_Y - 2E) \otimes \mathcal{O}_C \to \mathcal{O}_C$ at Q_2 is either $\mathcal{O}_C$ or $\mathcal{O}_C(-Q_2)$. $s(-1, -2) \geq -1$ by Lemma 4.7(i). As in the proof of (e) in Lemma 4.7(i) we can compute that $h^1(\mathcal{R}_{0,-1}^{1,2}) = d(0, -2) - 1$, whence $s_C(1, 2) = -1$ if $d(0, -2) = 1$. Thus when $b_1 = r_1 - 1$, the image at Q_2 is $\mathcal{O}_C$ or $\mathcal{O}_C(-Q_2)$ unless $d(0, -2) \geq 2$. Therefore Step 3 is deduced.

We shall investigate the case left in Step 3.

Step 4. The assumption in Theorem 4.1 holds in the case left in Step 3.

Recall that $3 \leq r_1 < r_2$. By (2.6) we have

$$d(1, -2) - d(0, -2) = -\frac{5}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \sum_{i=1,2}\left(\frac{2(r_i - 2)}{2r_i} - \frac{3(r_i - 3)}{2r_i}\right) = -1,$$

whence $d(1, -2) \geq 1$ by the assumption that $d(0, -2) \geq 2$. In particular $1 < a/n < 2$ by (2.9), and we have only to prove that $d(-1, 1) = 0$. Note that $b_2 \geq 3$ when $1 < a/n < 2$ by $nb_2 \equiv a$ modulo r_2 and $n < a < r_2$ in Lemma 3.3. By (2.6) we have

$$d(0, 1) - d(-1, 1) = \left(-\frac{a}{n} + \frac{1}{2}\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \sum_{i=1,2}\left(B_{r_i}(-b_i + 1) - B_{r_i}(-b_i)\right)$$

$$= -1 + \sum_{i=1,2}\left(b_i - \frac{a}{n}\right)\frac{1}{r_i}.$$

By $b_1 = r_1 - 1$ in the assumption and $d(0, 1) = 0$ in (2.6), the above equality means that

$$d(-1, 1) = \left(\frac{a}{n} + 1\right)\frac{1}{r_1} + \left(\frac{a}{n} - b_2\right)\frac{1}{r_2}.$$

The first term $(a/n + 1)/r_1$ is < 1 by $a/n < 2$ and $r_1 \geq 3$, and the second term $(a/n - b_2)/r_2$ is < 0 by $a/n < 2$ and $b_2 \geq 3$. Therefore $d(-1, 1) = 0$ and Step 4 follows.

In the case left in Step 3, the surface S has at worst Du Val singularities by Step 4. In particular, as in the argument before Table 3, its pre-image $S^\sharp$

on the index-one cover $Q_2^\sharp \in Y^\sharp$ also has a Du Val singularity at $Q_2^\sharp$. Thus $S^\sharp$ has multiplicity ≤ 2 at $Q_2^\sharp$, or equivalently, the defining function h_S of $S^\sharp$ in x_{21}, x_{22}, x_{23}, whose weight is b_2, has order ≤ 2. On the other hand, from $C \subset S$, the restriction of h_S to $C^\dagger$ by $(x_{21}, x_{22}, x_{23})|_{C^\dagger} = (0, t^{c_2/r_2}, t^{1-\overline{b_2 c_2}/r_2})$ is identically zero. By $c_2 < r_2 - \overline{b_2 c_2}$, such a function h_S exists only if $c_2 = 1$ or $b_2 = 1, 2$. By $b_2 \geq 3$ as seen in the proof of Step 4, we obtain that $c_2 = 1$ also in the case left in Step 3.   q.e.d.

In general it is difficult to control the denominator n of the discrepancy, but on our assumption we can narrow the possibilities down to one value.

Lemma 4.13. *Suppose that f is of No 15″ with $s_C(-1, 0) \neq 0$. Then $n = 2$, $a < r_1 < r_2$ and $b_i = (a + r_i)/2$ for $i = 1, 2$.*

Proof. Recall the notation that $g = \gcd(a, n)$ and $a = ga'$, $n = gn'$. We claim that $g \leq 2$. Suppose that $g \neq 1$ and consider $D_{n', -a'} = n' K_Y - a' E \equiv 0$. By the map $\mathcal{O}_Y(-D_{n', -a'})^{\otimes r_1 r_2} \otimes \mathcal{O}_Y(r_1 r_2 D_{n', -a'}) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $s_C(-n', a') = -\sum_{i=1,2} w_{Q_i}^C(n'b_i - a')/r_i$. On the other hand, $s_C(-n', a') = -1$ by Lemma 4.7(i), and $w_{Q_1}^C(n'b_1 - a') = \overline{d_{Q_1}^g b_1}$ and $w_{Q_2}^C(n'b_2 - a') = r_2 - \overline{d_{Q_2}^g b_2}$ by Lemma 4.12. Therefore $\overline{d_{Q_1}^g b_1}/r_1 = \overline{d_{Q_2}^g b_2}/r_2$. This equality and Lemma 3.5(ii) imply that $\overline{d_{Q_1}^g b_1}/r_1 = \overline{d_{Q_2}^g b_2}/r_2 = 1/2$, whence $g = 2$.

We have $(a/n)(H \cdot C) = (-K_Y \cdot C)$. Thus Lemma 4.12 implies that

$$(4.9) \qquad \left(\frac{a}{n} - b_1\right)\frac{1}{r_1} = \left(\frac{a}{n} - b_2\right)\frac{1}{r_2}.$$

By this equality and (4.2), we have $(a/n - b_i)/r_i \in \mathbb{Z}/2$. In particular $a'/n' - b_i \in \mathbb{Z} \cdot (g/2)$ by $g \mid r_i$ in Theorem 3.2. Therefore $n' \mid 2$ if $g = 1$, and $n' = 1$ if $g = 2$. Anyway $n = 2$ holds.

We then claim that $b_2 = (a + r_2)/2$, which completes the lemma by (4.9). Recall that $a < r_2$ by Lemma 3.3. If a is odd, then the relation $2b_2 \equiv a$ modulo r_2 implies that $2b_2 = a + r_2$. If a is even, then r_i for $i = 1, 2$ is also even by Theorem 3.2 and thus b_i is odd. We have $d_{Q_i}^2 = r_i/2$ by Lemma 3.5(ii). In particular $1 - e_{Q_2} a/2 \equiv r_2/2$ modulo r_2, whence $b_2 = (a + r_2)/2$ by $v_{Q_2} = 1$ and $a < r_2$.   q.e.d.

The general elephant theorem for f is derived as a corollary.

Corollary 4.14. *If f is of No 15″ with $s_C(-1, 0) \neq 0$, then the assumption in Theorem 4.1 holds.*

Proof. Note that $b_i = (a + r_i)/2$ for $i = 1, 2$ in Lemma 4.13 and thus $d(-1, 0) = 1$ by (4.2). For $0 \leq j \leq a/2$, by (2.6) we have

$$d(0, j) - d(-1, j) = \left(j - \frac{a+1}{2}\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + B_{-1,j} - B_{-1,j-1}.$$

By (2.9), $d(0, 0) = 1$ and $d(0, j) = 0$ for $1 \leq j \leq a/2$. Hence

$$d(-1, j) - d(-1, j+1) = \left(\frac{1}{r_1} + \frac{1}{r_2}\right) + B_{-1,j+1} - 2B_{-1,j} + B_{-1,j-1}$$

$$+ \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{for } 1 \leq j \leq a/2 - 1. \end{cases}$$

By the notation $u_i^j := b_i - j > 0$, the term $B_{-1,j+1} - 2B_{-1,j} + B_{-1,j-1}$ equals

$$\sum_{i=1,2} \left(\frac{(u_i^j - 1)(r_i - u_i^j + 1)}{2r_i} - 2\frac{u_i^j(r_i - u_i^j)}{2r_i} + \frac{(u_i^j + 1)(r_i - u_i^j - 1)}{2r_i} \right)$$

$$= -\sum_{i=1,2} \frac{1}{r_i}.$$

Therefore we obtain that $d(-1, j) = 0$ for $1 \le j \le a/2$.   q.e.d.

The following geometric statement is important in finding a component of $S \cap E$ intersecting H properly.

Lemma 4.15. *Suppose that f is of No $15''$ with $s_C(-1,0) \ne 0$. Then $H \cap E$ is irreducible but non-reduced.*

Proof. We write $[H \cap E] = s[C] + [F]$ cycle-theoretically, and set $(n/a)(1/r_1 + 1/r_2) = (H \cdot [H \cap E]) = s(H \cdot C) + t/r_1 + u/r_2$. By $(H \cdot C) = 1/r_1 - 1/r_2$ in Lemma 4.12, we obtain that

$$(4.10) \qquad \frac{s + t - n/a}{r_1} = \frac{s + n/a - u}{r_2}.$$

By $r_1 < r_2$ in Lemma 4.12, we have $(t, u) = (0,0)$, $(1,0)$ or $(0,1)$. By Corollary 2.7, if $[F] \ne 0$ then $F \cong \mathbb{P}^1$ and F intersects C exactly at one of Q_1 and Q_2, and $(H \cdot F) = 1/r_1$ or respectively $1/r_2$. In this case we have $a/n < 2$ by (4.10), whence $(a, n) = (3, 2)$ by Lemma 4.13. Let Q_i where $i = 1$ or 2 be the one which F passes through. Then $(H \cdot F) = 1/r_i$ and $(S \cdot F) = 3/2r_i$. Since any hidden non-Gorenstein point has index 2, the estimate $(S \cdot F) = 3/2r_i$ implies that r_i is an odd integer.

The equality (4.10) becomes

$$(3s + 2)r_1 = (3s + 1)r_2 \quad \text{if } (t, u) = (1,0),$$
$$(3s - 1)r_1 = (3s - 2)r_2 \quad \text{if } (t, u) = (0,1),$$

whence we can set

$$r_1 = (3s + 1)k, \quad r_2 = (3s + 2)k, \quad \text{if } (t, u) = (1,0),$$
$$r_1 = (3s - 2)k, \quad r_2 = (3s - 1)k, \quad \text{if } (t, u) = (0,1),$$

for some integer k. Here s is even and k is odd by $2 \nmid r_i$, whence $r_1 + r_2$ is odd. On the other hand, $(a/n)^2 r_1 r_2 E^3 = 3(r_1 + r_2)/2$ has to be an integer by Lemma 2.4, which is a contradiction. Hence $(H \cap E)_{\mathrm{red}}$ is irreducible.

Consider the map $\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_Y(K_Y) \otimes \mathcal{O}_C \to \mathcal{O}_C$. By the description in Lemma 4.12, we have $w_{Q_i}^C(b_i) + w_{Q_i}^C(-b_i) = r_i$ for $i = 1, 2$. Thus $s_C(-1, 0) + s_C(1, 0) = -2$, whence $s_C(1, 0) = -1$ by the assumption and Lemma 4.7(i). If $H \cap E$ is reduced, or equivalently $C = H \cap E$, then $s_C(1, 0) = -2$ by $h^1(\mathcal{R}_{0,-1}^{1,0}) = 1$ in Lemma 2.6. Therefore $H \cap E$ is non-reduced.   q.e.d.

Corollary 4.16. $r_1 \nmid 2r_2$.

Proof. Set $[H \cap E] = s[C]$ with $s \ge 2$ as in Lemma 4.15. Then $1/r_1 + 1/r_2 = (S \cdot [H \cap E]) = s(S \cdot C) = (sa/n)(1/r_1 - 1/r_2)$ by Lemma 4.12, whence $r_2/r_1 = (as + 2)/(as - 2)$. On the condition that $a > n = 2$ and $s \ge 2$, the value of r_2/r_1 is contained in $\mathbb{Z}/2$ only if $(a, s, r_2/r_1) = (3, 2, 2)$ or $(5, 2, 3/2)$.

In each case, either r_1 or r_2 is even, which contradicts that $b_i = (a + r_i)/2$ in Lemma 4.13. Hence the corollary is deduced.   q.e.d.

We recall that the strategy for the case No 16 is to choose C which intersects D properly. In No $15''$ we instead prove that $S \cap E$ has an irreducible component which intersects H properly.

Lemma 4.17. *Suppose that f is of No $15''$ with $s_C(-1, 0) \neq 0$. Then $S \cap E$ has an irreducible component which intersects H properly.*

Proof. By Corollary 4.14, the birational transform S on Y of a general elephant S_X of X is also a general elephant of Y and has at worst Du Val singularities. We examine the dual graph for the partial resolution $f_S \colon S \to S_X$. $(H \cap E)_{\mathrm{red}}$ is irreducible by Lemma 4.15. Suppose that $C = (H \cap E)_{\mathrm{red}} = (S \cap E)_{\mathrm{red}}$. Then C is the only exceptional curve of f_S. By Lemma 4.15 and $a/n > 1$, the cycle $[S \cap E]$ has the component $[C]$ with coefficient ≥ 3. Thus the coefficient of C in the fundamental cycle, given by that of C in $(f^*H_X)|_S = H|_S + E|_S$, is ≥ 4. Hence the dual graph of the exceptional curve of f_S is as follows.

$$\circ$$
$$\geq 4$$

By Table 2 and the list of dual graphs from Du Val singularities at the beginning of Section 5, P has to be of type $cE/2$ and C is the curve labelled F_3 in the dual graph for type E_7 in Section 5. Moreover, $[S \cap E] = 3[C]$ and thus $-(C^2)_S = -(E \cdot E \cdot S)/9 = (1/r_1 + 1/r_2)/9$. On the other hand, $-(F_3)^2 = 1/12$ by the dual graph. Therefore $(1/r_1 + 1/r_2)/9 = 1/12$, which contradicts that $3 \leq r_1 < r_2$ in Lemma 4.12.   q.e.d.

Lemma 4.17 means that $S \cap E$ has an irreducible component other than C. The next lemma decides $S \cap E$ cycle-theoretically.

Lemma 4.18. $[S \cap E] = [C] + [V]$ *cycle-theoretically, where $[V]$ consists of one or two $[\mathbb{P}^1]$. V intersects C at and only at Q_2.*

Proof. Write $[S \cap E] = x[C] + [V]$ cycle-theoretically and set $1/r_1 + 1/r_2 = (H \cdot [S \cap E]) = x(H \cdot C) + y/r_1 + z/r_2$. By Lemma 4.12, we obtain that
$$\frac{x - 1 + y}{r_1} = \frac{x + 1 - z}{r_2}.$$
We have $x \geq 1$ by $C \subset S$ in Lemma 4.12 and $(y, z) \neq (0, 0)$ by Lemma 4.17. Hence by $r_1 < r_2$ in Lemma 4.12, we have $(y, z) = (1, 0)$ or $(0, 1)$, or $(x, y, z) = (1, 0, 2)$. If $(x, y, z) = (1, 0, 2)$ then $[S \cap E] = [C] + [V]$ with $(H \cdot V) = 2/r_2$ and the lemma holds by Corollary 4.16. From now on we suppose that $(y, z) = (1, 0)$ or $(0, 1)$ and derive a contradiction.

By Corollary 4.16, if $(y, z) = (1, 0)$ or $(0, 1)$ then $V \cong \mathbb{P}^1$ and V intersects C exactly at one of Q_1 and Q_2, and $(H \cdot V) = 1/r_1$ or respectively $1/r_2$. Let Q_i where $i = 1$ or 2 be the one which V passes through. Then $(H \cdot V)_{Q_i} = (H \cdot V) = 1/r_i$. Since E is Cartier outside Q_1, Q_2, the isomorphism $V \cong \mathbb{P}^1$ implies that S is smooth along V outside Q_i, whence there exists no hidden non-Gorenstein point on V. Thus by the map $\mathcal{O}_Y(S)^{\otimes r_i} \otimes \mathcal{O}_Y(-r_i S) \otimes \mathcal{O}_V \to \mathcal{O}_V$, we have $r_i s_V(-1, 0) - r_i(S \cdot V) = -w_{Q_i}^V(b_i)$. Since $(S \cdot V) =$

$(a/2)(H \cdot V) = a/2r_i$, we obtain that $w_{Q_i}^V(b_i) \equiv a/2$ modulo r_i. On the other hand $(H \cdot V)_{Q_i} = 1/r_i$ implies that $w_{Q_i}^V(1) = 1$, whence $w_{Q_i}^V(b_i) = b_i$. This contradicts that $b_i = (a + r_i)/2$ in Lemma 4.13. q.e.d.

It is the time to complete Theorem 4.3 on the assumption that $a/n > 1$.

Theorem 4.19. *If f is of No 15″ with discrepancy $a/n > 1$, then Theorem 4.3 holds.*

Proof. Theorem 4.3(i) holds when $s_C(-1, 0) = 0$ thanks to Lemma 4.6. Here we consider the case $s_C(-1, 0) \neq 0$ and derive Theorem 4.3(ii). First we construct the coordinates in Lemma 4.12 using the surfaces H, E and S.

Step 1. *We may choose the coordinates in Lemma* 4.12 *so that the pre-images $E^\sharp$, $S^\sharp$ on $Q_1^\sharp \in Y^\sharp$ of E, S are given by respectively $x_{12} = 0$, $x_{13} = 0$, and the pre-images $H^\sharp$, $S^\sharp$ on $Q_2^\sharp \in Y^\sharp$ of H, S are given by respectively $x_{21} = 0$, $x_{23} = 0$.*

The curve C is the intersection of S and E near Q_1 by Lemma 4.18, whence we obtain the desired coordinates x_{11}, x_{12}, x_{13} by the description of $Q_1 \in C \subset Y$ in Lemma 4.12. We consider the coordinates x_{21}, x_{22}, x_{23}. We have $(H \cdot V)_{Q_2} = (H \cdot V) = 2/r_2$ by Lemma 4.18, whence $H^\sharp$ has multiplicity ≤ 2 at $Q_2^\sharp$. If $H^\sharp$ is not smooth at $Q_2^\sharp$, then $\mathrm{wt}\, x_{2i} + \mathrm{wt}\, x_{2j} \equiv 1$ modulo r_2 for some $i, j = 1$, 2 or 3 with $(i, j) \neq (2, 2)$ by $C \subset H$. This implies that either $b_2 \equiv 2$ or $2b_2 \equiv 1$ modulo r_2. However, this contradicts that $2 \leq a < r_1 < r_2$ and $b_2 = (a + r_2)/2$ in Lemma 4.13. Hence $H^\sharp$ is smooth at $Q_2^\sharp$, and we can choose x_{21} so that $H^\sharp$ at $Q_2^\sharp$ is given by $x_{21} = 0$. On the other hand, as in the argument before Table 3, $S^\sharp$ on $Q_2^\sharp \in Y^\sharp$ has a Du Val singularity at $Q_2^\sharp$, whence $S^\sharp$ has multiplicity ≤ 2 at $Q_2^\sharp$. In fact $S^\sharp$ is smooth at $Q_2^\sharp$ by the same argument as for $H^\sharp$. Therefore in order to complete Step 1, we have only to prove that $\mathrm{wt}\, x_{21} \not\equiv \mathrm{wt}\, x_{23}$ modulo r_2, but this also follows from the equality $b_2 = (a + r_2)/2$ in Lemma 4.13.

We have decided the cycle $[S \cap E]$ in Lemma 4.18. In order to obtain the dual graph for the partial resolution $f_S \colon S \to S_X$, we must study the cycle $[H \cap S]$.

Step 2. *$[H \cap S] = [C] + [B]$ cycle-theoretically, where B is the birational transform of the curve $H_X \cap S_X$. B intersects E at and only at Q_1.*

Let c_{HS}, c_{HE}, c_{SE} denote the coefficients of $[C]$ in the cycles $[H \cap S]$, $[H \cap E]$, $[S \cap E]$. Consider the relation $(H \cap S) \cap (H \cap E) \subset S \cap E$. H is smooth at the generic point of C according to Step 1. Thus we obtain that $\min\{c_{HS}, c_{HE}\} \leq c_{SE}$. This implies that $c_{HS} = 1$ since $c_{HE} > 1$ by Lemma 4.15 and $c_{SE} = 1$ by Lemma 4.18. Hence the equality $[H \cap S] = [C] + [B]$ holds. Then $(E \cdot B) = (E \cdot [H \cap S]) - (E \cdot C) = 2/r_1$, whence B intersects E at and only at Q_1 by Corollary 4.16.

The coefficient of C in the fundamental cycle is given by that of C in $(f^*H_X)|_S = H|_S + E|_S$, and this is two by Lemma 4.18 and Step 2. By Lemma 4.18 and Steps 1, 2, we obtain the dual graph for f_S described in Theorem 4.3(ii). Thanks to Lemma 4.13 and Corollary 4.14, it suffices to prove that either (a) or (b) in Theorem 4.3(ii) holds.

Step 3. *Either* (a) *or* (b) *in Theorem* 4.3(ii) *holds.*

$r_2 - r_1 = 2(b_2 - b_1)$ by Lemma 4.13. Set $r_2 - r_1 = 2k$. Note that if a is even then each b_i is odd and thus k is even. We also set $[H \cap E] = s[C]$ as in Lemma 4.17. Then $1/r_1 + 1/r_2 = (S \cdot [H \cap E]) = s(S \cdot C) = (sa/2)(1/r_1 - 1/r_2)$ by Lemma 4.12, or equivalently $r_1 + r_2 = sak$. Thus
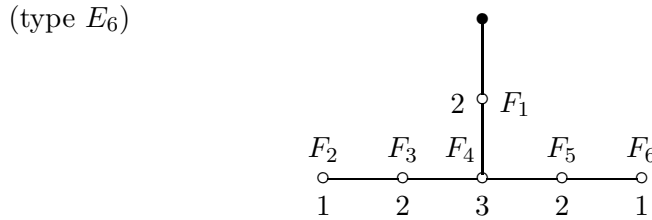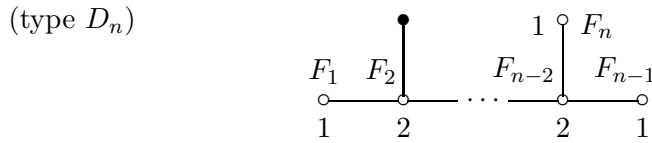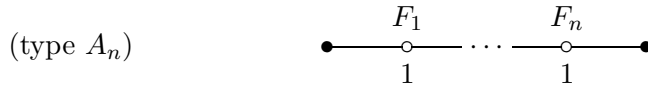
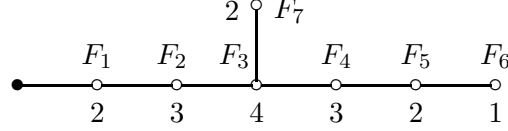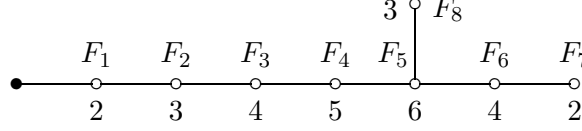$$(4.11) \qquad r_1 = (as/2 - 1)k, \qquad r_2 = (as/2 + 1)k.$$

On the other hand, by using the coordinates in Step 2, we can express $H^\sharp$ on $Q_1^\sharp \in Y^\sharp$ as $x_{12} \cdot (\text{something}) + x_{13}^s = 0$, and $E^\sharp$ on $Q_2^\sharp \in Y^\sharp$ as $x_{21} \cdot (\text{something}) + x_{23}^s = 0$. Considering weights, we obtain that $b_1 s \equiv 1$ modulo r_1 and $b_2 s \equiv -1$ modulo r_2. By $b_i = (a + r_i)/2$, this implies that $as \equiv 2$ modulo r_1 and $as \equiv -2$ modulo r_2. Therefore $(as, k) = (2r_1 + 2, 1)$ or $(r_1 + 2, 2)$ by (4.11). These cases correspond to respectively (a), (b) in Theorem 4.3(ii). This completes Step 3, and thus the theorem.　　q.e.d.

5. Case of small discrepancies

In this section we study the cases left in Sections 3 and 4 by using the general elephant theorem in the strongest form. In the former half, we proceed with the discussion on the co-primeness of a and n after Section 3 and complete Theorem 1.3. We also construct examples of divisorial contractions in the cases in the first, second, third and last lines of Table 1. In the latter half, we prove Theorem 4.3 when f is of general type with discrepancy $a/n \le 1$.

According to Theorem 4.1 and Corollary 4.2, we can start from taking a general elephant of S as the birational transform of a general elephant S_X of X. It is reasonable to examine the crepant morphism $f_S \colon S \to S_X$ in terms of the dual graph as in Theorem 4.3. Hence we here recall the dual graph for the minimal resolution of a Du Val singularity. ∘ denotes an exceptional curve and ● the birational transform of a general hyperplane section, and each exceptional curve F_i is marked with its coefficient in the fundamental cycle. Note that S_X has a Du Val singularity of type at worst E_7 at P if X is not Gorenstein by Table 3.

(type A_n)

(type D_n)

(type E_6)

(type E_7)

$$2 \overset{\circ}{} F_7$$

$F_1 \quad F_2 \quad F_3 \quad\quad F_4 \quad F_5 \quad F_6$

●———○———○———○———○———○———○

$2 \qquad 3 \qquad 4 \qquad 3 \qquad 2 \qquad 1$

(type E_8)

$$3 \overset{\circ}{} F_8$$

$F_1 \quad F_2 \quad F_3 \quad F_4 \quad F_5 \quad\quad F_6 \quad F_7$

●———○———○———○———○———○———○

$2 \qquad 3 \qquad 4 \qquad 5 \qquad 6 \qquad 4 \qquad 2$

Throughout this section, we assume that either f belongs to one of the cases left in Theorem 3.7 or f is of general type with discrepancy $a/n \leq 1$ for $a \neq 1$. We let H_X be a general hyperplane section on the germ $P \in X$, and H the birational transform on Y of H_X. We analyse the intersections $H \cap E$ and $S \cap E$ to obtain the dual graph for $f_S \colon S \to S_X$. In our case we have $f^* S_X = S + (a/n)E$ since S gives a general elephant of Y, whence $S \cap E$ has no embedded points and $(S \cap E)_{\mathrm{red}}$ is a tree of $\mathbb{P}^1$ by Corollary 2.7. On the other hand, by Lemma 3.8 and the equality (4.1), we obtain that $f^* H_X = H + E$ except for the case No 8 where $f^* H_X = H + 2E$. Anyway $H \cap E$ has no embedded points and $(H \cap E)_{\mathrm{red}}$ is also a tree of $\mathbb{P}^1$.

We proceed to study the cases left in Theorem 3.7. We use the notation and data in Lemma 3.8 freely. By Theorems 3.7, 5.1 and 5.5, we complete Theorem 1.3. First we treat the case No 8.

Theorem 5.1. *If f is of No 8, then a is co-prime to n except for the case where P is of type $cE/2$ and $(a, n) = (2, 2)$. This exception is placed in the last line of Table 1.*

Proof. We suppose the case left in Theorem 3.7. Since $(-E \cdot [S \cap E]) = E^3 = 1/6$ and $6E$ is Cartier, we obtain that $S \cap E \cong \mathbb{P}^1$ scheme-theoretically. In particular S is smooth outside Q and Q'. We let C denote this irreducible reduced curve $S \cap E$. E is smooth at the generic point of C, and the 1-cycle $2[C]$ defines a member of $\mathbb{P}H^0(\mathcal{Q}_{-2,0})$. By $H \sim 2S$, the 1-cycle $[H \cap E]$ also defines a member of $\mathbb{P}H^0(\mathcal{Q}_{-2,0})$. Thus we obtain that $[H \cap E] = 2[C]$ since $h^0(\mathcal{Q}_{-2,0}) = d(-2, 0) = d(0, -2) = 1$ by (2.7) and Lemma 3.8. Hence the coefficient of C in the fundamental cycle of the Du Val singularity $P \in S_X$, given by that of C in $(f^* H_X)|_S = H|_S + 2E|_S$, is ≥ 3, and the dual graph of the exceptional curve of $f_S \colon S \to S_X$ is as follows.

$$\overset{\circ}{\geq 3}$$

Therefore P is of type $cE/2$ by Table 3, and the theorem is deduced. q.e.d.

There exists an example of f of No 8 with discrepancy 1.

Example 5.2. Let $P \in X$ be the germ

$$o \in (x_1^2 + x_4^3 + x_2 x_3^3 x_4 + x_2^4 + x_3^8 = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4} / \frac{1}{2}(1, 1, 1, 0).$$

P is of type $cE/2$. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 2, 1, 3)$. Then f is a divisorial contraction of No 8 with discrepancy 1.

We treat the cases No 14 and 15′a simultaneously. If f is No 14 then Q is a quotient singularity by $r \geq 6$. If f is of No 15′a then Q is of type cA/r by $r \geq 4$, and a general elephant of the germ $Q \in Y$ is A_{2r-1} by [24].

Lemma 5.3. *If a is not co-prime to n in No 14 or 15′a, then*

(i) *S_X has a Du Val singularity of type D_k at P, where $k \geq r$ in No 14 and $k \geq 2r$ in No 15′a.*

(ii) *S has a Du Val singularity at Q of type A_{r-1} in No 14, and of type A_{2r-1} in No 15′a.*

(iii) *The set-theoretic intersection of H, E, S is the point Q.*

Proof. We suppose the case left in Theorem 3.7. If S_X has a Du Val singularity of type A at P, then H intersects $S \cap E$ properly at two points by the dual graph for f_S. Then $4/r = (H \cdot [S \cap E]) \geq 1 + 2/r$ in No 14 and $2/r = (H \cdot [S \cap E]) \geq 1 + 1/r$ in No 15′a, which is a contradiction. Hence S_X has a Du Val singularity of type D or E_7 at P by $n = 2$. In No 14 $(H \cdot [S \cap E]) = a'E^3 = 2/r'$ and $r'H$ is Cartier, and in No 15′a $(H \cdot [S \cap E]) = a'E^3 = 2/r$ and rH is Cartier. Thus the 1-cycle $[S \cap E]$ consists of at most two irreducible reduced cycles. We use the trichotomy that $S \cap E$ is either irreducible and reduced, reduced but reducible, or irreducible but non-reduced. In the following special case, we assume that $S \cap E$ is reduced.

Step 1. We may assume that $S \cap E$ is reduced if f is of No 14 with $(a, r) = (2, 6)$ or $(4, 10)$.

We have the surjective map $f_* \mathcal{O}_Y(S) \twoheadrightarrow H^0(\mathcal{Q}_{-1,0})$ by Lemma 2.6. Thus if $h^0(\mathcal{Q}_{-1,0}) \geq 2$ then the restriction on E of the linear system $|S|$ moves, and we can choose S so that $S \cap E$ is reduced. Since $h^0(\mathcal{Q}_{-1,0}) = d(-1, 0)$ by (2.7), the estimate $h^0(\mathcal{Q}_{-1,0}) \geq 2$ holds if and only if $d(0,0) - d(-1, 0) \leq -1$ by (2.9). Computing $d(0, 0) - d(-1, 0)$ by (2.6) with Lemma 3.8, we obtain Step 1.

First we prove (iii).

Step 2. The set-theoretic intersection of H, E, S is the point Q.

By $(H \cdot E \cdot S) < 1$, it suffices to prove that H intersects $S \cap E$ properly. If $S \cap E$ is non-reduced and if $(S \cap E)_{\mathrm{red}} \subset H$, then the dual graph of the exceptional curve of f_S is as follows.

$$ Q \xrightarrow{} \circ $$
$$ \geq 3 $$

S has a Du Val singularity at Q of type A_{r-1} or worse in No 14, and of type A_{2r-1} or worse in No 15′a. By Step 1 and Lemma 3.8, we have $r \geq 10$ in No 14 and $r \geq 4$ in No 15′a, whence $Q \in S$ is A_7 or worse. However, there exists no possibility for the dual graph to be described as above. If $S \cap E$ is reduced and if $S \cap E \subset H$, then $a = 2$ and $H \cap E = S \cap E$ scheme-theoretically. Thus 0 is linearly equivalent to the restriction of $H - S \sim K_Y - E$ on E, whence 0 can be regarded as a member of $\mathbb{P}H^0(\mathcal{Q}_{1,-1})$. This contradicts that $h^0(\mathcal{Q}_{1,-1}) = d(1, -1) = 0$ in (2.7) and (2.9).

To complete Step 2, we need to exclude the case where $S \cap E$ is reducible and H contains one irreducible component of $S \cap E$. In this case the dual

graph of the exceptional curves of f_S is as follows.

$$\underset{1}{\circ}\!\!\!-\!\!\!-\!\!\!-\; Q \;-\!\!\!-\!\!\!-\!\!\!\underset{\geq 2}{\circ}$$

S has a Du Val singularity at Q of type A_{r-1} or worse in No 14, and of type A_{2r-1} or worse in No 15′a. S is smooth outside Q since E is Cartier outside Q. Hence S_X has a Du Val singularity of type E_7 at P and the exceptional divisor $E|_S$ of f_S has to be $F_1 + F_6$ or $F_1 + F_7$. If $E|_S = F_1 + F_6$ then $-(E|_S)^2 = -(F_1 + F_6)^2 = 3/4$, which contradicts that $-(E|_S)^2 = 4/r$ in No 14 and $2/r$ in No 15′a. If $E|_S = F_1 + F_7$ then $(E|_S \cdot F_7) = 0$, which contradicts the f-ampleness of $-E$. Therefore Step 2 follows.

The statement (ii) follows from (iii).

Step 3. S has a Du Val singularity at Q of type A_{r-1} in No 14, and of type A_{2r-1} in No 15′a.

On the index-one cover $Q^\sharp \in Y^\sharp$, the local intersection number of the pre-images is $(H^\sharp \cdot E^\sharp \cdot S^\sharp)_{Q^\sharp} = 4$ in No 14 and 2 in No 15′a. Hence if f is of No 14 then $S^\sharp$ is smooth at $Q^\sharp$, because $H^\sharp$ and $E^\sharp$ are not smooth at $Q^\sharp$. Similarly, if f is of No 15′a then $S^\sharp$ is the restriction of a smooth divisor on the tangent space $T_{Q^\sharp}Y^\sharp = \mathbb{C}^4$ of $Y^\sharp$ at $Q^\sharp$. Now Step 3 is deduced from the explicit description of the terminal singularity $Q \in Y$. Here we should remark the a priori property that S has a Du Val singularity at Q.

By Step 2, we can describe the dual graph for f_S as follows according to the trichotomy.

(i) $S \cap E$ is irreducible and reduced.

$$\bullet\!\!\!-\!\!\!-\!\!\!-\; Q \;-\!\!\!-\!\!\!-\!\!\!\underset{1}{\circ}$$

(ii) $S \cap E$ is reduced but reducible.

$$\begin{array}{c}\overset{\circ}{\,\,}\;1\\ \big|\\ \bullet\!\!\!-\!\!\!-\!\!\!-\; \underset{1}{Q} \;-\!\!\!-\!\!\!-\!\!\!\underset{1}{\circ}\end{array}$$

(iii) $S \cap E$ is irreducible but non-reduced.

$$\bullet\!\!\!-\!\!\!-\!\!\!-\; Q \;-\!\!\!-\!\!\!-\!\!\!\underset{2}{\circ}$$

The type of the Du Val singularity $Q \in S$ is described in Step 3. If $S \cap E$ is reduced then S is smooth outside Q, since E is Cartier outside Q.

Step 4. S_X has a Du Val singularity of type D at P.

If otherwise then S_X has a Du Val singularity of type E_7 at P. S has a Du Val singularity of type A_k at Q with $k \geq 5$ by the estimate of r' in Lemma 3.8, and if $S \cap E$ is non-reduced then $k \geq 7$ by Step 1. However, there exists no possibility for the dual graph to be described as above, whence Step 4 is deduced.

According to Step 3, the number of the exceptional curves in the minimal resolution of $P \in S_X$ is $\geq r$ in No 14 and $\geq 2r$ in No $15'$a. Therefore the statement (i) follows, and the lemma is completed.   q.e.d.

In particular by Lemma 5.3(i) and Table 3, P is of type $cAx/2$ or $cD/2$. We consider the morphism $f' \colon (Y' \supset E') \to (X' \ni P')$, the double cover of f, constructed in (3.1). We set $\alpha_X \colon X' \to X$ and $\alpha_Y \colon Y' \to Y$ as in (3.1), and set $Q' := \alpha_Y^{-1}(Q)$, $H' := \alpha_Y^* H$, $S' := \alpha_Y^* S$. Let T' be a general member of the linear system $|-K_{Y'}| = |S'| = |a'H'|$, and $T'_{X'}$ its birational transform on X'. The surface T' is a general elephant of Y', and the induced morphism $f_{T'} \colon T' \to T'_{X'}$ is a crepant morphism between normal surfaces with Du Val singularities. We shall examine the type of the Du Val singularity $P' \in T'_{X'}$.

Lemma 5.4. $T'_{X'}$ *has a Du Val singularity of type D at P'. Consequently P is of type $cD/2$.*

Proof. We claim that both $Q' \in S'$ and $Q' \in T'$ are of type $A_{r'-1}$ in No 14, and of type $A_{2r'-1}$ in No $15'$a. Indeed by Lemma 5.3(ii), the class group of the germ $Q \in S$ is $\mathbb{Z}/(r)$ in No 14 and $\mathbb{Z}/(2r)$ in No $15'$a, and thus the claim on $Q' \in S'$ follows. By $S' \sim a'H' \sim -K_{Y'}$, we can regard S' as a specialisation of T', whence $P' \in T'$ is at worst $A_{r'-1}$ in No 14, and at worst $A_{2r'-1}$ in No $15'$a. On the other hand by Lemma 3.1, the non-Gorenstein point $Q' \in Y'$ generates one, respectively two fictitious singularities in No 14, respectively No $15'$a. Hence the claim on $Q' \in T'$ follows from the classification of three-fold terminal singularities.

We have known that P is of type $cAx/2$ or $cD/2$. Suppose that $T'_{X'}$ has a Du Val singularity of type A at P'. Equivalently, P is of type $cAx/2$ and $P' \in S'_{X'} := \alpha_X^*(S_X)$ is of type A. Then $T' \cap E'$ must intersect a general member of $|H'|$ properly at two points, whence $(H' \cdot T' \cdot E') \geq 1 + 1/r'$. However, $(H' \cdot T' \cdot E') = 2(H \cdot S \cdot E) = 4/r'$ in No 14 and $2/r'$ in No $15'$a, whence f has to be of No 14 with $r' = 3$ by Lemma 3.8. In this case by Step 1 in Lemma 5.3, $S \cap E$ is reduced and S' has the only singularity Q', which is A_2. The number of the irreducible components of $S' \cap E'$ is ≤ 4. Hence we obtain that $P' \in S'_{X'}$ is at worst A_6 by counting the number of the exceptional curves in the minimal resolution of $P' \in S'_{X'}$. On the other hand, $P \in S_X$ is D_6 or worse by Lemma 5.3(i). This contradicts Table 3. Therefore $T'_{X'}$ has a Du Val singularity of type D at P', and P is of type $cD/2$.   q.e.d.

We can restrict the possibility of the case $(a, n) = (4, 2)$ as follows.

Theorem 5.5. *If f is of No 14 or $15'$a, then a is co-prime to n except for the case where P is of type $cD/2$ and $(a, n) = (2, 2)$ or $(4, 2)$. The case $(a, n) = (4, 2)$ happens only if f is of No 14 with $r' \equiv 1$ or 7 modulo 8. These exceptions are placed in the first, second and third lines of Table 1.*

Proof. By Theorem 3.7 and Lemmata 3.8, 5.4, it suffices to prove the claim on the case $(a, n) = (4, 2)$. Take any irreducible reduced component $C \cong \mathbb{P}^1$ of $H \cap E$, and set $C' := \alpha_Y^{-1}(C)_{\mathrm{red}}$. If C' is irreducible then by Lemma 5.3(iii) the induced morphism $C' \to C$ of degree 2 is ramified at and only at Q, which contradicts the formula $2\chi(\mathcal{O}_{C'}) = 2 \cdot 2\chi(\mathcal{O}_C) - 1$ of Hurwitz. Hence C splits into two $\mathbb{P}^1$ by the covering α_Y. Write $[H \cap E] = \sum_i c_i[C_i]$

cycle-theoretically. For each $C_i \cong \mathbb{P}^1$ we choose one irreducible component C_i^1 of $\alpha_Y^{-1}(C_i)_{\mathrm{red}}$, and construct a 1-cycle $C^1 := \sum_i c_i[C_i^1]$ on Y'. If f is of No $15'$a with $(a, n) = (4, 2)$, then $(H' \cdot C^1) = 1/2(H' \cdot H' \cdot E') = 1/2r'$. This contradicts that $r'H'$ is Cartier. Hence the case $(a, n) = (4, 2)$ happens only if f is of No 14. In this case $(H' \cdot C^1) = 1/r'$, whence $C^1 = [\mathbb{P}^1]$ and thus $H \cap E = \mathbb{P}^1$.

Suppose that f is of No 14 with $(a, n) = (4, 2)$, and consider a normal form of $Q \in C := H \cap E \subset Y$. Q is a quotient singularity of type $\frac{1}{r}(1, -1, r' + 4)$. S intersects C properly at Q with $(S \cdot C)_Q = 4/r$ by Lemma 5.3(iii). Hence the restriction of the defining function of its pre-image $S^\sharp$ on $Q^\sharp \in Y^\sharp$, whose weight is $r' + 4$, to $C^\dagger$ has order $4/r$ with respect to t. Thus $w_Q^C(r' + 4) = 4$. Since $r' \geq 5$ is an odd integer and $(r', 0) \in G_Q^C$, we may choose semi-invariant local coordinates x_1, x_2, x_3 with weights $\mathrm{wt}(x_1, x_2, x_3) = (1, -1, r' + 4)$ of the index-one cover $Q^\sharp \in Y^\sharp$ so that $(x_1, x_2, x_3)|_{C^\dagger} = (t^{(r'+1)/r}, t^{(r'-1)/r}, t^{4/r})$, and so that $S^\sharp$ is given by $x_3 = 0$. The defining functions h_H, h_E of $H^\sharp$, $E^\sharp$ have weights 2, -2 respectively. By $(H \cdot E \cdot S)_Q = 4/r$, the monomials x_1^2, x_2^2 appear in respectively h_H, h_E with non-zero coefficients. On the other hand by $C \subset H, E$, the restrictions $h_H|_{C^\dagger}$ and $h_E|_{C^\dagger}$ are identically zero. For the terms $x_1^2|_{C^\dagger} = t^{2(r'+1)/r}$ in $h_H|_{C^\dagger}$ and $x_2^2|_{C^\dagger} = t^{2(r'-1)/r}$ in $h_E|_{C^\dagger}$ to be cancelled out, both of the following must hold.

(i) One of the monomials $x_1 x_3^{(r'+1)/4}$, $x_2 x_3^{(r'+3)/4}$, $x_3^{(r'+1)/2}$ appears in h_H with non-zero coefficient. Their weights are respectively $2 + r'(r' + 5)/4$, $2 + r'(r' + 7)/4$, $2 + r'(r' + 5)/2$. Hence $r' \equiv 3$ modulo 8, $r' \equiv 1$ modulo 8, or $r' \equiv 3$ modulo 4.

(ii) One of the monomials $x_1 x_3^{(r'-3)/4}$, $x_2 x_3^{(r'-1)/4}$, $x_3^{(r'-1)/2}$ appears in h_E with non-zero coefficient. Their weights are respectively $-2 + r'(r' + 1)/4$, $-2 + r'(r' + 3)/4$, $-2 + r'(r' + 3)/2$. Hence $r' \equiv 7$ modulo 8, $r' \equiv 5$ modulo 8, or $r' \equiv 1$ modulo 4.

Therefore $r' \equiv 1$ or 7 modulo 8, and we complete the theorem. q.e.d.

There exist examples of f of No 14 or $15'$a with discrepancy 1.

Example 5.6. Let $P \in X$ be the germ

$$o \in \left(\begin{array}{c} x_1^2 + x_4 x_5 + x_3^{r'+1} = 0 \\ x_2^2 + x_3^{r'-1} + x_4^{r'-1} + x_5 = 0 \end{array} \right) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4 x_5}/\frac{1}{2}(1, 1, 1, 0, 0),$$

where r' is odd with $r' \neq 1$. P is of type $cD/2$. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (\frac{r'+1}{2}, \frac{r'-1}{2}, 1, 1, r')$. Then f is a divisorial contraction with discrepancy 1. f is of No 14 with $J = \{(2r', 2)\}$.

Example 5.7. (i) Let $P \in X$ be the germ

$$o \in (x_1^2 + x_2 x_3 x_4 + x_2^4 + x_3^4 + x_4^4 = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{2}(1, 1, 1, 0).$$

P is of type $cD/2$. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (2, 2, 1, 1)$. Then f is a divisorial contraction with discrepancy 1. f is of No $15'$a with $J = \{(4, 1), (4, 1)\}$.

(ii) Let $P \in X$ be the germ

$$o \in (x_1^2 + x_2^2 x_4 + x_3^{2r'} + x_4^{2r'} = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{2}(1,1,1,0),$$

where r' is even. P is of type $cD/2$. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r', r', 1, 1)$. Then f is a divisorial contraction with discrepancy 1. f is of No $15'$a with $J = \{(2r',1),(2r',1)\}$.

There also exist examples of f of No 14 with discrepancy 2.

Example 5.8.   (i) Let $P \in X$ be the germ

$$o \in \left(\begin{array}{l} x_1^2 + x_4 x_5 + x_2 x_3^{(r'+3)/4} = 0 \\ x_2^2 + x_3^{(r'-1)/2} + x_4^{r'-1} + x_5 = 0 \end{array} \right) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4 x_5}/\frac{1}{2}(1,1,1,0,0),$$

where $r' \equiv 1$ modulo 8 with $r' \neq 1$. P is of type $cD/2$. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (\frac{r'+1}{2}, \frac{r'-1}{2}, 2, 1, r')$. Then f is a divisorial contraction with discrepancy 2. f is of No 14 with $J = \{(2r',2)\}$.

(ii) Let $P \in X$ be the germ

$$o \in \left(\begin{array}{l} x_1^2 + x_4 x_5 + x_3^{(r'+1)/2} = 0 \\ x_2^2 + x_1 x_3^{(r'-3)/4} + x_4^{r'-1} + x_5 = 0 \end{array} \right) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4 x_5}/\frac{1}{2}(1,1,1,0,0),$$

where $r' \equiv 7$ modulo 8. P is of type $cD/2$. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (\frac{r'+1}{2}, \frac{r'-1}{2}, 2, 1, r')$. Then f is a divisorial contraction with discrepancy 2. f is of No 14 with $J = \{(2r',2)\}$.

The rest of this section is devoted to the proof of Theorem 4.3 when f is of general type with discrepancy $a/n \leq 1$. We keep the notation given just before Theorem 4.3. If S_X has a Du Val singularity of type A at P, then H intersects $S \cap E$ properly at two points by the dual graph for f_S, and consequently Theorem 4.3(i) holds by Lemma 4.4(ii). Hence we assume that S_X has a Du Val singularity of type D or E at P. In particular $n = 2$, 3 or 4. We shall describe the partial resolution $f_S \colon S \to S_X$.

Lemma 5.9. *Suppose that f is of general type with discrepancy $a/n \leq 1$ for $a \neq 1$, and that S_X has a Du Val singularity of type D or E at P. Then the dual graph of the exceptional curves of f_S is as follows.*

(i) *f is of No $15''$.*
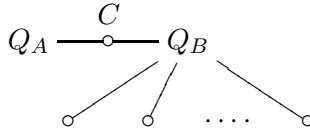
The curve C is contained in $H|_S$. $(A,B) = (1,2)$ or $(2,1)$ and S has Du Val singularities of types $A_{r_A - 1}$, $A_{r_B - 1}$ or worse respectively at Q_A, Q_B. S has no singularities other than Q_1 and Q_2 on any exceptional curve at the generic point of which $S \cap E$ is smooth.

(ii) *f is of No 16.*
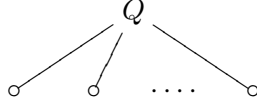
> *S has a Du Val singularity of type A_{r-1} or worse at Q. There exists an exceptional curve at the generic point of which $S \cap E$ is smooth, and on any such curve, S has no singularities other than Q. If • intersects an irreducible reduced exceptional curve at a point other than Q, then this curve is contained in $H|_S$.*

Proof. First we consider the case No 15″. We claim that $S \cap E$ has an irreducible component which passes through Q_1 and Q_2. If otherwise then $S \cap E$ is a tree of $Q_1 \in C_1 \cong \mathbb{P}^1$ and $Q_2 \in C_2 \cong \mathbb{P}^1$ by $(H \cdot S \cdot E) = 1/r_1 + 1/r_2$. Hence $S \cap E \subset H$ by Lemma 4.4(ii). This could happen only if $a/n = 1$ and $H \cap E = S \cap E$ scheme-theoretically. However in this case, 0 is linearly equivalent to the restriction of $H - S \sim K_Y - E$ on E, because E is smooth at the generic points of C_1 and C_2. Thus 0 can be regarded as a member of $\mathbb{P}H^0(\mathcal{Q}_{1,-1})$, which contradicts that $h^0(\mathcal{Q}_{1,-1}) = d(1,-1) = 0$ in (2.7) and (2.9). Hence $S \cap E$ has an irreducible component C which passes through Q_1 and Q_2. Such a curve C is uniquely determined because $(S \cap E)_{\mathrm{red}}$ is a tree of $\mathbb{P}^1$. C is contained in H by Lemma 4.4(ii), whence we obtain the desired dual graph by $(H \cdot S \cdot E) = 1/r_1 + 1/r_2$, remarking that H and E are Cartier outside Q_1, Q_2.

We then consider the case No 16. In this case $(a,n) = (2,3)$ or $(3,4)$ by the co-primeness of a and n in Theorem 3.2, and $r \geq a + n$ by Lemma 3.3. By the surjective map $f_*\mathcal{O}_Y(S) \twoheadrightarrow H^0(\mathcal{Q}_{-1,0})$ in Lemma 2.6 and the estimate $h^0(\mathcal{Q}_{-1,0}) = 2 + a/n + (a/n - b)/r \geq 2$ in (4.2), there exists an irreducible component of $S \cap E$ at the generic point of which $S \cap E$ is smooth. The desired dual graph is obtained by Lemma 4.4(ii) provided that any exceptional curve passes through Q.

Suppose that there exists an exceptional curve not passing through Q. Then $S \cap E$ is a tree of $Q \in C_1 \cong \mathbb{P}^1$ and $Q \notin C_2 \cong \mathbb{P}^1$ with a node Q' by $(H \cdot S \cdot E) = 1 + 1/r$. In particular $(H \cdot C_1) = 1/r$, and S is smooth outside Q, Q' since E is Cartier outside Q. Hence Q' is a hidden non-Gorenstein point of index n by $(S \cdot C_1) = a/nr$. We have $C_1 \subset H$ by Lemma 4.4(ii). Thus we obtain the following dual graph of the exceptional curves of f_S.

$$Q \overset{C_1}{\underset{\geq 2}{\rule{1.5cm}{0.4pt}}}\!\!\circ\ Q' \overset{C_2}{\underset{\geq 1}{\rule{1.5cm}{0.4pt}}}\!\!\circ$$
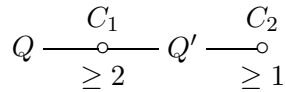
S has Du Val singularities of types A_{r-1}, A_{n-1} or worse respectively at Q, Q', and they are all the singularities of S. If $(a,n) = (2,3)$ then $P \in S_X$ is of type E_6 by Table 3, but there exists no possibility for the dual graph to be described as above. If $(a,n) = (3,4)$ then $P \in S_X$ is of type D_k by Table 3, and $(C_1, C_2) = (F_l, F_1)$ for $5 \leq l \leq k - 3$ or (F_l, F_k) for $r \leq l \leq k - 4$ up to permutation. If $(C_1, C_2) = (F_l, F_1)$ then $E|_S = F_1 + F_l$ and $(E|_S \cdot C_1) = (F_1 + F_l \cdot F_l) = 0$, which is a contradiction. If $(C_1, C_2) = (F_l, F_k)$ then $E|_S = F_l + F_k$ and $-(E|_S)^2 = -(F_l + F_k)^2 = 1/l + 1/(k - l) \leq 1/r + 1/4$,

which also contradicts that $-(E|_S)^2 = -(E \cdot E \cdot S) = 1 + 1/r$. Therefore any exceptional curve passes through Q, and the lemma is deduced.   q.e.d.

We shall narrow down the possibilities of f_S by using Lemma 5.9.

Lemma 5.10. *If f is of general type with discrepancy $a/n \le 1$ for $a \ne 1$, then S_X has a Du Val singularity of type A or D at P.*

Proof. Indeed if S_X has a Du Val singularity of type E_7, then $(a, n) = (2, 2)$ and thus f is of No 15″ by Theorem 3.2. Moreover, $2 \mid r_1, r_2$ by Theorem 3.2 and $4 \mid r_1, r_2$ by $e_{Q_1} = r_1/2 + 1$, $e_{Q_2} = r_2/2 + 1$ in Lemma 3.5(ii). However, there exists no possibility for the dual graph to be described as in Lemma 5.9.

If S_X has a Du Val singularity of type E_6, then $(a, n) = (2, 3)$ or $(3, 3)$ and thus $r_2 \ge 5$ by Lemma 3.3. If f is of No 15″ then $E|_S$ has to be F_3 up to permutation by Lemma 5.9. In this case $r_1 = 2$, $r_2 = 5$ and $-(E|_S)^2 = 1/r_1 + 1/r_2 = 7/10$, which contradicts that $-(F_3)^2 = 3/10$. On the other hand, if f is of No 16 then $E|_S$ has to be either F_1, F_2, $F_1 + F_2$ or $F_2 + F_6$ up to permutation by Lemma 5.9. Then $-(E|_S)^2$ is respectively $-(F_1)^2 = 1/2$, $-(F_2)^2 = 3/4$, $-(F_1 + F_2)^2 = 4/5$, $-(F_2 + F_6)^2 = 1$, which contradicts that $-(E|_S)^2 = 1 + 1/r$.   q.e.d.

Therefore for Theorem 4.3, the case where $P \in S_X$ is of type D remains. We list all the possibilities of the exceptional locus of f_S in Tables 7 and 8 up to permutation by Lemma 5.9.

TABLE 7. f is of No 15″ and $P \in S_X$ is of type D_k.

$E|_S$	$-(E|_S)^2$	r_A	r_B
$F_1 + F_l,\ 2 \le l \le k - 3$	1		$\le l - 1$
$F_l,\ 2 \le l \le k - 3$	$1/l$	$\le l$	
$F_l + 2F_m,\ 2 \le l < m \le k - 2$	$1/l + 1/(m - l)$	$\le l$	$\le m - l$
$2F_l + F_m,\ 2 \le l < m \le k - 3$	$4/l + 1/(m - l)$		$\le m - l$
$F_l + F_k,\ 2 \le l \le k - 3$	$1/l + 1/(k - l)$	$\le l$	$\le k - l$
$F_l + F_{k-1} + F_k,\ 2 \le l \le k - 3$	$1/l + 1/(k - l - 1)$	$\le l$	$\le k - l - 1$

TABLE 8. f is of No 16 and P is of type D_k.

$E|_S$	$-(E|_S)^2$	r
F_1	1	
$F_1 + 2F_l,\ 2 \le l \le k - 2$	$1 + 1/(l - 1)$	$\le l - 1$
$F_1 + F_k$	$1 + 1/(k - 1)$	$\le k - 1$
$F_1 + F_{k-1} + F_k$	$1 + 1/(k - 2)$	$\le k - 2$
F_k	$4/k$	$\le k$
$F_{k-1} + F_k$	$4/(k - 1)$	$\le k - 1$

In Table 7, the case $E|_S = F_1 + F_l$ does not occur by $(E|_S \cdot F_l) = 0$, the case $E|_S = F_l$ does not occur by $-(E|_S)^2 = 1/r_1 + 1/r_2$, and the case $E|_S = 2F_l + F_m$ does not occur by $(E|_S \cdot F_m) = 1/(m - l) > 0$. If f is of No 16 and if $P \in S_X$ is of type D, then $(a, n) = (3, 4)$ by Theorem 3.2 and

$r \geq 7$ by Lemma 3.3. Hence the cases $E|_S = F_1$, F_k, $F_{k-1} + F_k$ in Table 8 do not occur by $-(E|_S)^2 = 1 + 1/r$. By this observation and the equality $-(E|_S)^2 = 1/r_1 + 1/r_2$ with Tables 7 and 8, we obtain the following lemma. Here the curve C in Lemma 5.11 stands for F_l in Table 7 for No 15″, and F_1 in Table 8 for No 16.

Lemma 5.11. *Theorem 4.3 holds except for the case where f satisfies the following conditions.*

(i) *S_X has a Du Val singularity of type D at P. In particular $n = 2$ or 4.*

(ii) *S has a Du Val singularity of type A_{r_Q-1} at each $Q \in I$.*

(iii) *$[S \cap E] = [C] + [V]$ cycle-theoretically, where $[V]$ consists of one or two $[\mathbb{P}^1]$. V intersects C at and only at Q_2 in No 15″ and Q in No 16.*

(iv) *$[S \cap H] = [C] + [B]$ in No 15″ and $[S \cap H] = [B]$ in No 16 cycle-theoretically, where B is the birational transform on S of $H_X|_{S_X}$. B intersects E at and only at Q_1 in No 15″ and Q in No 16, except for the case $r_1 = 2$ in No 15″.*

(v) *$(H \cdot C) = 1/r_1 - 1/r_2$. In particular $r_1 < r_2$.*

It is the time to complete Theorem 4.3.

Theorem 5.12. *If f is of general type with discrepancy $a/n \leq 1$, then Theorem 4.3 holds.*

Proof. It suffices to study the case left in Lemma 5.11. We derive Theorem 4.3(ii) in this case. Since $S \cap E$ and thus S are smooth along C outside the singularities in I, there exists no hidden non-Gorenstein point on C. We first exclude the case No 16. If f is of No 16 then $(a, n) = (3, 4)$ by Theorem 3.2.

Step 1. f is of No 15″. If $(a, n) = (3, 4)$ then V is non-reduced.

Indeed, if $(a, n) = (3, 4)$ then we have the divisor $D_{1,-1} = K_Y - E \equiv (1/4)H$ and $(D_{1,-1} \cdot V) = 1/2r_2$. Hence V has to pass through a hidden non-Gorenstein point, say Q'. In particular V is non-reduced since V is the intersection of S and E outside C. We shall exclude the case No 16. If f is of No 16 then $(D_{1,-1} \cdot C) = (r - 1)/4r$, whence $(r - 1)/4$ is an integer. On the other hand, also $(r+1)/4$ must be an integer by the equality $h^0(\mathcal{Q}_{-1,0}) = 2 + (3(r+1)/4 - b)/r$ in (4.2). This is a contradiction, and Step 1 is deduced.

For the case No 15″, we shall construct good semi-invariant local coordinates x_{i1}, x_{i2}, x_{i3} with weights $\mathrm{wt}(x_{i1}, x_{i2}, x_{i3}) = (1, -1, b_i)$ of the index-one cover $Q_i^\sharp \in Y^\sharp$ at each non-Gorenstein point Q_i on C for $i = 1, 2$.

Step 2. We may choose the coordinates x_{i1}, x_{i2}, x_{i3} so that

$$Q_i^\sharp \in C^\sharp \subset Y^\sharp \cong o \in (x_{ii}\text{-axis}) \subset \mathbb{C}^3_{x_{i1}x_{i2}x_{i3}}.$$

Moreover, the pre-images $E^\sharp$, $S^\sharp$ on $Q_1^\sharp \in Y^\sharp$ of E, S are given by respectively $x_{12} = 0$, $x_{13} = 0$, and the pre-images $H^\sharp$, $S^\sharp$ on $Q_2^\sharp \in Y^\sharp$ of H, S are given by respectively $x_{21} = 0$, $x_{23} = 0$.

S has a Du Val singularity of type A_{r_i-1} at each Q_i. Thus $S^\sharp$ is smooth at $Q_i^\sharp$ since the class group of the germ $Q_i \in S$ is $\mathbb{Z}/(r_i)$. Hence we may choose the coordinates so that $Q_i^\sharp \in S^\sharp$ is given by $x_{i3} = 0$ for each i. The equality $(H \cdot V)_{Q_2} = 2/r_2$ implies that $H^\sharp$ has multiplicity ≤ 2 at $Q_2^\sharp$, and $H^\sharp$ has multiplicity 2 only if $V \cong \mathbb{P}^1$. If $H^\sharp$ is not smooth at $Q_2^\sharp$, then $(a, n) \neq (3, 4)$ by Step 1. Hence a and n has a common divisor 2 and thus r_2 is even by Theorem 3.2. This contradicts the equality $\mathrm{wt}\, x_{2i} + \mathrm{wt}\, x_{2j} \equiv 1$ modulo r_2 for some $i, j = 1$, 2 or 3. Thus $H^\sharp$ is smooth at $Q_2^\sharp$. In order to obtain the coordinates x_{21}, x_{22}, x_{23} in Step 2, we have only to prove that $b_2 \neq 1$. Since $nb_2 \equiv a$ modulo r_2 and $r_2 > r_1 \geq 2$, the equality $b_2 = 1$ holds only if $(a, n) = (2, 2)$ or $(4, 4)$. However if $(a, n) = (2, 2)$ or $(4, 4)$ and $b_2 = 1$, then $e_2 = 1$ and $d_2^n = 0$, which contradicts Lemma 3.5(ii). Therefore $b_2 \neq 1$ and we obtain the desired coordinates x_{21}, x_{22}, x_{23}. In particular, $w_{Q_2}^C(1) = r_2 - 1$.

By the map $\mathcal{O}_Y(H)^{\otimes r_1 r_2} \otimes \mathcal{O}_Y(-r_1 r_2 H) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $s_C(0, -1) - (H \cdot C) = -w_{Q_1}^C(1)/r_1 - w_{Q_2}^C(1)/r_2$. By $w_{Q_2}^C(1) = r_2 - 1$ and $(H \cdot C) = 1/r_1 - 1/r_2$, we obtain that $w_{Q_1}^C(1) \equiv 1$ modulo r_1. Hence we may choose the coordinates x_{11}, x_{12}, x_{13} which satisfy all the conditions in Step 2 but the condition that $E^\sharp$ is given by $x_{12} = 0$ at $Q_1^\sharp$. However, we can obtain this condition because C is the intersection of S and E near Q_1 and its pre-image $C^\sharp$ at $Q_1^\sharp$ is given by the x_{11}-axis.

Step 3. $(a, n) = (2, 2)$. $4 \mid r_i$ and $b_i = r_i/2 + 1$ for each i.

By the map $\mathcal{O}_Y(S)^{\otimes r_1 r_2} \otimes \mathcal{O}_Y(-r_1 r_2 S) \otimes \mathcal{O}_C \to \mathcal{O}_C$, we have $s_C(-1, 0) - (S \cdot C) = -w_{Q_1}^C(b_1)/r_1 - w_{Q_2}^C(b_2)/r_2$. Since $w_{Q_1}^C(b_1) = b_1$ and $w_{Q_2}^C(b_2) = r_2 - b_2$ by Step 2, we have $(S \cdot C) = b_1/r_1 - b_2/r_2 + s_C(-1, 0) + 1$. On the other hand $(S \cdot C) = (a/n)(H \cdot C) = (a/n)(1/r_1 - 1/r_2)$ by Lemma 5.11, whence $(a/n - b_1)/r_1 \equiv (a/n - b_2)/r_2$ modulo $\mathbb{Z}$. Thus we obtain that $(a, n) = (2, 2)$ as in the proof of Lemma 4.13. The rest of Step 3 follow from $e_i = r_i/2 + 1$ in Lemma 3.5(ii).

By Lemma 5.11 and the above steps, it remains only proving (b) in Theorem 4.3(ii). However the proof of Step 3 in Theorem 4.19 is valid also in our situation. Therefore the theorem is completed.   q.e.d.

6. DIVISORIAL CONTRACTIONS TO POINTS OF TYPE cA/n

In this section we prove the ordinary case (i) of Theorem 1.1. Our strategy for the classification is to determine E in the sense of valuation by applying the next lemma, which is a reformulation of [14, Lemma 6.1]. We recall that for a proper birational morphism $Z \to X$, the *centre* of E on Z is the locus $\pi_Z(\pi_{Y*}^{-1}E)$ for a common resolution of singularities W of Y and Z with $\pi_Y \colon W \to Y$, $\pi_Z \colon W \to Z$.

Lemma 6.1. *Let* $f \colon (Y \supset E) \to (X \ni P)$ *be a germ of a three-fold divisorial contraction to a point* P *of index* n *identified with*

$$P \in X \cong o \in (\phi = 0) \subset \bar{X} := \mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{n}(\mathrm{wt}\, x_1, \mathrm{wt}\, x_2, \mathrm{wt}\, x_3, \mathrm{wt}\, x_4).$$

*Let a/n denote the discrepancy of f, and m_i/n the multiplicity of x_i along E.
Suppose that $(m_1/n, m_2/n, m_3/n, m_4/n)$ is primitive in the group $N := \mathbb{Z}^4 + \mathbb{Z}(\mathrm{wt}\, x_1/n, \mathrm{wt}\, x_2/n, \mathrm{wt}\, x_3/n, \mathrm{wt}\, x_4/n)$. Let d/n denote the weighted order of
ϕ with respect to weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1/n, m_2/n, m_3/n, m_4/n)$,
and decompose ϕ as*

$$\phi = \phi_{d/n}(x_1, x_2, x_3, x_4) + \phi_{>d/n}(x_1, x_2, x_3, x_4),$$

*where $\phi_{d/n}$ is the weighted homogeneous part of weight d/n and $\phi_{>d/n}$ is the
part of weight $> d/n$. Set $c/n := m_1/n + m_2/n + m_3/n + m_4/n - 1 - d/n$.
Let $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} \ni P)$ be the weighted blow-up of $\bar{X}$ with weights
$\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1/n, m_2/n, m_3/n, m_4/n)$, $\bar{F}$ its exceptional divisor,
and $\bar{D}_i$ the birational transform on $\bar{Z}$ of the $\mathbb{Q}$-Cartier divisor $x_i = 0$ on
$\bar{X}$. Let Z denote the birational transform on $\bar{Z}$ of X, and $g\colon Z \to X$ the
induced morphism. Suppose that $\bar{F} \cap Z$ defines an irreducible reduced 2-cycle
$[F]$ on Z. We suppose that Z is smooth at the generic point of F, and that
$\dim(\mathrm{Sing}\, \bar{Z} \cap Z) \le 1$. Let L denote the centre of E on Z. Then,*

- *(i) $L \not\subseteq \bigcup_{1 \le i \le 4} \bar{D}_i$, and F has multiplicity 1 along E.*
- *(ii) If $L \ne F$ and Z is canonical at the generic point of L, then $c/n < a/n$.*
- *(iii) If $L = F$, then $c/n = a/n$ and $f \cong g$ over X.*

Proof. Z is normal by the criterion of Serre, and F is a $\mathbb{Q}$-Cartier anti-f-ample divisor. Let M and M_i for $1 \le i \le 4$ denote the multiplicities along E of F and $\bar{D}_i|_Z$. Then $m_i = Mm_i + M_i$ for each i. On the other hand, at least one M_i equals zero by $\bigcap_{1 \le i \le 4} \bar{D}_i = \emptyset$, whence $M = 1$ and $M_i = 0$ for all i. This implies (i). The rest (ii) and (iii) follow from $M = 1$, the adjunction formula $K_Z = g^* K_X + (c/n)F$ and [12, Lemma 3.4]. q.e.d.

The morphisms $\bar{g}$ and g in Lemma 6.1 are described in terms of toric geometry. In the situation of Lemma 6.1, we identify $P \in \bar{X}$ with the germ of the toric variety corresponding to the fan generated by $\vec{e}_1 := (1, 0, 0, 0)$, $\ldots$, $\vec{e}_4 := (0, 0, 0, 1)$ in N, and $\bar{f}$ with the operation of adding the ray generated by the primitive vector $\vec{e} := (m_1/n, m_2/n, m_3/n, m_4/n)$. Then $\bar{Y}$ is covered by four charts $\bar{U}_i$ for $1 \le i \le 4$, where $\bar{U}_i$ corresponds to the fan generated by $\vec{e}$ and all $\vec{e}_j$ with $j \ne i$. Note that $\bar{U}_i = \bar{Z} \setminus \bar{D}_i$.

We proceed to the classification in the cA/n case. From now on, we assume that $f\colon (Y \supset E) \to (X \ni P)$ is a divisorial contraction to a point of type cA/n with $n \ge 2$. The classification of f has been obtained when P is a terminal quotient singularity by Kawamata in [17], and when the discrepancy a/n is $1/n$ by Hayakawa in [10, Theorem 6.4]. We can immediately verify that Theorem 1.1(i) holds in these cases. Therefore in this section we assume that $a \ge 2$ and that P is not a quotient singularity. Then in particular f is of general type by Theorem 1.3 and Corollary 2.5, and thus we can use Theorem 4.3, the general elephant theorem in strong form. The dual graph belongs to the case (i) of Theorem 4.3.

We follow the notation in Section 4. We let H_X be a general hyperplane section on the germ $P \in X$ and H its birational transform on Y, let S be a general elephant of Y and S_X its birational transform on X. Then $f^* H_X = H + E$ by (4.1), and $f^* S_X = S + (a/n)E$. According to Theorem

4.3(i), the surfaces H, E, S intersect properly at and only at two points Q_1 and Q_2 of Y. Q_1, Q_2 correspond to the points Q_1, Q_2 in I in No 15″, and to a smooth point and Q in I in No 16. We let r_1, r_2 denote the local indices of Y at Q_1, Q_2. By $(H \cdot E \cdot S) = (a/n)E^3 = 1/r_1 + 1/r_2$ in Table 4, we have the local intersection number $(H \cdot E \cdot S)_{Q_i} = 1/r_i$ for each i. In particular we may choose semi-invariant local coordinates x_{i1}, x_{i2}, x_{i3} of the index-one cover $Q_i^\sharp \in Y^\sharp$ of the germ $Q_i \in Y$ so that the pre-images $H^\sharp$, $E^\sharp$, $S^\sharp$ on $Q_i^\sharp \in Y^\sharp$ of H, E, S are given by

$$(6.1) \qquad H^\sharp = (x_{i1} = 0), \qquad E^\sharp = (x_{i2} = 0), \qquad S^\sharp = (x_{i3} = 0).$$

We let h_{aH_X}, h_{nS_X} denote the defining functions on X of aH_X, nS_X respectively. We can regard h_{aH_X} and h_{nS_X} as elements in the vector space $V_a := f_* \mathcal{O}_Y(-aE)/f_* \mathcal{O}_Y(-(a+1)E)$. The following lemma plays a pivotal role in applying Lemma 6.1.

Lemma 6.2. h_{aH_X} and h_{nS_X} are linearly independent in V_a.

Proof. The statement means that any function $h := c_1 h_{aH_X} + c_2 h_{nS_X}$ for $c_1, c_2 \in \mathbb{C}$ with $(c_1, c_2) \neq (0,0)$ has multiplicity a along E. By the description (6.1), on the germ $Q_i \in Y$ the function h_{aH_X}, h_{nS_X} are expressed as $h_{aH_X} = u_1 x_{i1}^a x_{i2}^a$ and $h_{nS_X} = u_2 x_{i3}^n x_{i2}^a$, where u_1, u_2 are units. Then h is expressed as $h = (c_1 u_1 x_{i1}^a + c_2 u_2 x_{i3}^n) x_{i2}^a$, whence h vanishes with multiplicity a along $x_{i2} = 0$, that is, along E. Thus the lemma is deduced.   q.e.d.

We shall choose a good identification of $P \in X$ in Theorem 2.2.

Lemma 6.3. *We may choose the identification*

$$(6.2) \quad P \in X \cong o \in (\phi := x_1 x_2 + g(x_3^n, x_4) = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{n}(1, -1, b, 0)$$

so that $h_{nS_X} \equiv x_3^n$, $h_{aH_X} \equiv x_4^a$ modulo $f_* \mathcal{O}_Y(-(a+1)E)$.

Proof. Take the index-one cover $P^\sharp \in X^\sharp$ of $P \in X$, and let h_{S_X}, h_{H_X} denote the defining functions of the pre-images of S_X, H_X. The lemma follows from standard argument on coordinates change. Since S_X has a Du Val singularity of type A at P, we can take the identification (6.2) so that $h_{S_X} = x_3 + $ (non-linear terms), see [24, Corollary 4] for example. Since H_X is a general hyperplane section on X, we can set $h_{H_X} \equiv x_4 + x_1 p(x_1, x_3) + q(x_2, x_3)$ modulo $f_* \mathcal{O}_Y(-2E)$. By making coordinates change in which $x_4 + x_1 p \mapsto x_4$, we eliminate x_1 from h_{H_X} modulo $f_* \mathcal{O}_Y(-2E)$. The same argument yields the coordinates in which $h_{H_X} \equiv x_4$ modulo $f_* \mathcal{O}_Y(-2E)$.

We have preserved the coordinate x_3 in the above argument, whence $h_{S_X} = x_3 + $ (non-linear terms). Suppose that we can write $h_{H_S} \equiv x_3 + u_1 x_1 p(x_1, x_2, x_4) + u_2 q(x_2, x_4)$ modulo $f_* \mathcal{O}_Y(-K_Y - E)$, where u_1, u_2 are units. Then by the coordinates change in which $x_3 + u_1 x_1 p \mapsto x_3$, we obtain a new identification (6.2) in which $h_{H_S} \equiv v_1 x_3 + x_1^{n-1} p^n r(x_1, x_2, x_4) + u_2 q(x_2, x_4)$ modulo $f_* \mathcal{O}_Y(-K_Y - E)$, where v_1 is a unit. Repeating this argument, we can eventually obtain the desired coordinates.   q.e.d.

We proceed to apply Lemma 6.1 to f. Let m_i/n denote the multiplicity of x_i in Lemma 6.3 along E. Then $m_3/n = a/n$ and $m_4/n = 1$. We distribute the weights by $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1/n, m_2/n, a/n, 1)$. For a

semi-invariant function h, we let w-ord h denote the weighted order of h with respect to these weights. Set $D :=$ w-ord ϕ and decompose ϕ as $\phi = \phi_D + \phi_{>D}$ as in Lemma 6.1.

Lemma 6.4. w-ord $g = m_1/n + m_2/n$. *Consequently $D = m_1/n + m_2/n$ and $\phi_D = x_1 x_2 + g_D(x_3^n, x_4)$, where g_D is the weighted homogeneous part of weighted order D in g.*

Proof. Set $D_0 :=$ w-ord g and let g_{D_0} denote the weighted homogeneous part of weighted order D_0 in g. It suffices to prove that $D_0 = m_1/n + m_2/n$. If $D_0 > m_1/n + m_2/n$ then $x_1 x_2 = -g_{D_0} \in f_* \mathcal{O}_Y(-D_0 E)$, which is a contradiction. Hence $D_0 \leq m_1/n + m_2/n$, and it suffices to prove that $g_{D_0} \notin f_* \mathcal{O}_Y(-(D_0 + 1)E)$. This g_{D_0} always decomposes into a product of the monomial $x_4^{D_0 - a \lfloor D_0/a \rfloor}$ and $\lfloor D_0/a \rfloor$ linear combinations of the monomials x_3^n and x_4^a. The function x_4 has multiplicity 1 along E, and any function defined by a non-zero linear combination of x_3^n and x_4^a has multiplicity a by Lemmata 6.2 and 6.3. Hence g_{D_0} has multiplicity D_0 along E, and thus the lemma is deduced. q.e.d.

By using Lemma 6.4, we shall verify the assumption of Lemma 6.1 for f. Set $N := \mathbb{Z}^4 + \mathbb{Z}(1/n, -1/n, b/n, 0)$ and $\vec{e} := (m_1/n, m_2/n, a/n, 1)$. It is obvious that the exceptional divisor F of the weighted blow-up g constructed in Lemma 6.1 is irreducible and reduced.

(i) N is generated by $\vec{e}, \vec{e}_2, \vec{e}_3, \vec{e}_4$ and
$$(-1/n, 1/n, -b/n, 0) = \frac{1}{m_1}\left(-\vec{e} + \frac{m_1 + m_2}{n}\vec{e}_2 + \frac{-bm_1 + a}{n}\vec{e}_3 + \vec{e}_4\right).$$
We see that
$$\bar{U}_1 = \mathbb{C}^4_{x_1' x_2' x_3' x_4'} / \frac{1}{m_1}\left(-1, \frac{m_1 + m_2}{n}, \frac{-bm_1 + a}{n}, 1\right),$$
$$(x_1, x_2, x_3, x_4) = (x_1'^{m_1/n}, x_1'^{m_2/n} x_2', x_1'^{a/n} x_3', x_1' x_4').$$
In particular Sing $\bar{Z} \subseteq (x_1' = x_4' = 0)$. $Z|_{\bar{U}_1}$ is given by
$$x_2' + g_D(x_3'^n, x_4') + x_1' \cdot (\text{something}) = 0,$$
whence Sing $Z|_{\bar{U}_1}$ is contained in
$$(x_1' = x_4' = x_2' + g_D(x_3'^n, 0) = 0).$$

(ii) $\bar{U}_2 = \mathbb{C}^4_{x_1' x_2' x_3' x_4'} / \frac{1}{m_2}(\frac{m_1 + m_2}{n}, -1, \frac{bm_2 + a}{n}, 1)$. Sing $Z|_{\bar{U}_2}$ is contained in
$$(x_2' = x_4' = x_1' + g_D(x_3'^n, 0) = 0).$$

(iii) N is generated by $\vec{e}_1, \vec{e}_2, \vec{e}, \vec{e}_4$ and $(-b'/n, b'/n, -1/n, 0)$ for an integer b' with $bb' \equiv 1$ modulo n. $\bar{U}_3 = \mathbb{C}^4_{x_1' x_2' x_3' x_4'} / \frac{1}{a}(\frac{-b'a + m_1}{n}, \frac{b'a + m_2}{n}, -1, 1)$. Sing $Z|_{\bar{U}_3}$ is contained in
$$\{x_4'\text{-axis}\} \cup (x_3' = x_4' = x_1' x_2' + g_D(1, 0) = 0).$$

(iv) $\bar{U}_4 = \mathbb{C}^4_{x_1' x_2' x_3' x_4'} / \frac{1}{n}(1, -1, a, 0)$. Sing $Z|_{\bar{U}_4}$ is contained in x_3'-axis.

Therefore the assumption of Lemma 6.1 holds, and Sing $Z \subseteq \bigcup_{1 \leq i \leq 4} \bar{D}_i$. Since $c/n := (m_1/n + m_2/n + a/n + 1) - 1 - D = a/n$ by Lemma 6.4, we can deduce the following theorem from Lemma 6.1.

Theorem 6.5. $f \cong g$ over X. Hence f is the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1/n, m_2/n, a/n, 1)$.

Therefore $Z \cong Y$ must be terminal. Z has a quotient singularity of type $\frac{1}{m_1}(-1, \frac{-bm_1+a}{n}, 1)$ at the origin of the chart $\bar{U}_1$, and has a quotient singularity of type $\frac{1}{m_2}(-1, \frac{bm_2+a}{n}, 1)$ at the origin of the chart $\bar{U}_2$. Hence m_1 is co-prime to $(-bm_1 + a)/n$, and m_2 is co-prime to $(bm_2 + a)/n$. On this condition, Z is terminal on the chart $\bar{U}_3$ only if Z does not pass through the origin of $\bar{U}_3$, or equivalently $g_D(1,0) \neq 0$. Now it is obvious that f can be expressed as in Theorem 1.1(i). We can also verify that any f in Theorem 1.1(i) is a divisorial contraction by using Lemma 6.6. Therefore we complete Theorem 1.1(i).

Lemma 6.6. Let a, n, b, b', r_1, r_2 be integers which satisfy the following conditions.

(i) $bb' \equiv 1$, $a \equiv br_1$ modulo n, and $r_1 + r_2 \equiv 0$ modulo an.
(ii) $(a - br_1)/n$ is co-prime to r_1.

Then $(a+br_2)/n$ is co-prime to r_2, and $(r_1 - b'a)/n$, $(r_2 + b'a)/n$ are co-prime to a.

Proof. Set $g := \gcd(a, n)$, $a = ga'$, $n = gn'$. Then $ga' \equiv br_1$ modulo gn'. Since b is co-prime to $n = gn'$, we have $g \mid r_1$ and thus $g \mid r_2$. Set $r_1 = gr_1'$, $r_2 = gr_2'$. Then $(a' - br_1')/n'$ is co-prime to gr_1', whence $\gcd(a' - br_1', r_1') \mid n'$. This implies that a' is co-prime to r_1', and thus a' is co-prime to r_2' by $r_1' + r_2' \equiv 0$ modulo $a'n'g$. On the other hand by $g \mid -b(r_1' + r_2')/n' = (a' - br_1')/n' - (a' + br_2')/n'$, we obtain that $(a' + br_2')/n'$ is also co-prime to g. Therefore $(a+br_2)/n = (a'+br_2')/n'$ is co-prime to $r_2 = gr_2'$. We have seen that a' is co-prime to r_1'. Since $b(r_1' - b'a')/n' \equiv (br_1' - a')/n'$ modulo g, we obtain that $(r_1' - b'a')/n'$ is co-prime to g. Hence $(r_1 - b'a)/n = (r_1' - b'a')/n'$ is co-prime to $a = ga'$. Similarly, $(r_2 + b'a)/n$ is co-prime to a.   q.e.d.

7. THE EXCEPTIONAL CASE OF TYPE cD/2

We have obtained Theorem 1.3 in Sections 3 and 5, and this implies Theorem 1.1(ii) in the case where f is of exceptional type by Corollary 2.5. We have proved Theorem 1.1(i) in Section 6. In this section we study the case (ii) of Theorem 4.3 to complete the theorems stated in Section 1. We need to prove that in this case P is of type cD/2 and f is described as in Theorem 1.2. It should be emphasised that whichever type in Table 4 f belongs to, we can apply the method of explicit description of f below by formal modification. Throughout this section, we keep assuming that f belongs to the case (ii) of Theorem 4.3. In particular P is of type cAx/2 or cD/2 by Table 3. We first provide examples of Theorem 1.2.

Example 7.1.   (i) Let $P \in X$ be the germ

$$o \in (x_1^2 + x_2^2 x_4 + x_3^{2(r+1)/a} + x_4^{r+1} = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{2}(1, 1, 1, 0),$$

where $a \mid r+1$ and a, r are odd with $r \neq 1$. P is of type cD/2. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = ((r + 2)/2, r/2, a/2, 1)$. Then f is a divisorial contraction with discrepancy $a/2$. f is of No 15″ with $J = \{(r, 1), (r + 2, 1)\}$.

(ii) Let $P \in X$ be the germ

$$o \in \left(\begin{array}{c} x_1^2 + x_2 x_5 + x_4^{r+2} = 0 \\ x_2 x_4 + x_3^{(r+2)/a} + x_5 = 0 \end{array} \right) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4 x_5}/\frac{1}{2}(1,1,1,0,1),$$

where $a \mid r+2$ and $(r+2)/a$ is odd with $r \neq 1$ and $a \neq r+2$. P is of type $cD/2$. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = ((r+2)/2, r/2, a/2, 1, (r+4)/2)$. Then f is a divisorial contraction with discrepancy $a/2$. f is of No $15''$ with $J = \{(r,1), (r+4,1)\}$.

We start with computing the dimensions of the vector spaces

$$V^{[0]}(i) := f_* \mathcal{O}_Y(-iE)/f_* \mathcal{O}_Y(-(i+1)E) \qquad\qquad (i \in \mathbb{Z}),$$
$$V^{[1]}(i) := f_* \mathcal{O}_Y(K_Y - (i + a/2)E)$$
$$\qquad\qquad / f_* \mathcal{O}_Y(K_Y - (i + 1 + a/2)E) \qquad (i + a/2 \in \mathbb{Z}).$$

Define $V^{[j]}(i)$ for every $i \in \mathbb{Z}/2$ by setting $V^{[j]}(i) := 0$ if $i + ja/2 \notin \mathbb{Z}$. The vector space $V^{[j]}(i)$ is regarded as the space of semi-invariant functions on X with multiplicity i along E and with weight j. We set

$$V(i) := V^{[0]}(i) \oplus V^{[1]}(i).$$

We define the finite set

$$N_k(i) := \left\{ \vec{l} = (l_1, l_2, l_3, l_4) \in \mathbb{Z}_{\geq 0}^4 \;\Big|\; \frac{r_1 + k}{2} l_1 + \frac{r_1}{2} l_2 + \frac{a}{2} l_3 + l_4 = i, \; l_1 l_2 = 0 \right\}$$

for $i \in \mathbb{Z}/2$ and $k = 2, 4$, and decompose it as

$$N_k(i) = N_k^{[0]}(i) \sqcup N_k^{[1]}(i),$$

where $N_k^{[j]}(i) := \{\vec{l} \in N_k(i) \mid l_1 + l_2 + l_3 \equiv j \text{ modulo } 2\}$. Note that $N_k^{[j]}(i) = \emptyset$ when $i + ja/2 \notin \mathbb{Z}$ by $a \equiv r_1$ modulo 2.

Lemma 7.2. $\dim V^{[j]}(i) = \# N_{r_2 - r_1}^{[j]}(i)$.

Proof. We have $\dim V^{[j]}(i) = d(j, -i - ja/2)$ for $i \geq -a/2$ with $i + ja/2 \in \mathbb{Z}$ by (2.8), and $jb_{Q_k} - (i + ja/2)v_{Q_k} \equiv -(i + jr_k/2)$ modulo r_k for $k = 1, 2$ by Theorem 4.3(ii). Set $s_k := i + jr_k/2$. Thus by (2.6), for any $i \geq 0$ with $i + ja/2 \in \mathbb{Z}$,

$$\dim V^{[1-j]}(i - a/2) - \dim V^{[j]}(i)$$
$$= -\left(i + \frac{1}{2}\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \sum_{k=1,2} \left(\frac{\overline{s_k}(r_k - \overline{s_k})}{2 r_k} - \frac{(\overline{s_k} + 1)(r_k - \overline{s_k} - 1)}{2 r_k} \right)$$
$$= j - 1 - \sum_{k=1,2} \frac{s_k - \overline{s_k}}{r_k}.$$

Hence for any $i \geq 0$ with $i + ja/2 \in \mathbb{Z}$,

$$(7.1) \qquad \dim V^{[j]}(i) - \dim V^{[1-j]}(i - a/2) = 1 - j + \sum_{k=1,2} \left\lfloor \frac{i}{r_k} + \frac{j}{2} \right\rfloor.$$

On the other hand, we have a decomposition

$$N^{[j]}_{r_2-r_1}(i) = \{(l_1, l_2, l_3+1, l_4) \mid \vec{l} \in N^{[1-j]}_{r_2-r_1}(i-a/2)\}$$
$$\sqcup \{\vec{l} \in N^{[j]}_{r_2-r_1}(i) \mid l_1, l_3 = 0\}$$
$$\sqcup \{\vec{l} \in N^{[j]}_{r_2-r_1}(i) \mid l_2, l_3 = 0, \ l_1 \geq 1\}.$$

Hence for any $i \geq 0$ with $i + ja/2 \in \mathbb{Z}$,

$$(7.2) \quad \#N^{[j]}_{r_2-r_1}(i) - \#N^{[1-j]}_{r_2-r_1}(i - a/2)$$
$$= \#\{\vec{l} \in N^{[j]}_{r_2-r_1}(i) \mid l_1, l_3 = 0\} + \#\{\vec{l} \in N^{[j]}_{r_2-r_1}(i) \mid l_2, l_3 = 0, \ l_1 \geq 1\}$$
$$= \left(1 + \left\lfloor \frac{i - jr_1/2}{r_1} \right\rfloor\right) + \left(1 + \left\lfloor \frac{i - (2-j)r_2/2}{r_2} \right\rfloor\right)$$
$$= 1 - j + \sum_{k=1,2} \left\lfloor \frac{i}{r_k} + \frac{j}{2} \right\rfloor.$$

The lemma holds trivially if $i < 0$ or $i + ja/2 \notin \mathbb{Z}$. Therefore the lemma follows from (7.1) and (7.2).   q.e.d.

We shall find bases of the vector space $V(i)$ in terms of any identification

$$P \in X \cong o \in (\phi = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4} / \frac{1}{2}(1,1,1,0).$$

For a semi-invariant function h, we let $\mathrm{mult}_E\, h$ denote the multiplicity of h along E.

Lemma 7.3.   (i) $\mathrm{mult}_E\, x_4 = 1$ and $\mathrm{mult}_E\, x_i \geq a/2$ for any $i = 1, 2, 3$. There exists some $k = 1, 2, 3$ with $\mathrm{mult}_E\, x_k = a/2$. By permutation we may assume that $x_k = x_3$.

(ii) The monomials x_3^2 and x_4^a are linearly independent in $V^{[0]}(a)$. For $i < r_1/2$, the monomials $x_3^{l_3} x_4^{l_4}$ for $(0, 0, l_3, l_4) \in N_{r_2-r_1}(i)$ form bases of $V(i)$. In particular for $k = 1, 2$, $\mathrm{mult}_E\, \bar{x}_k \geq r_1/2$ for $\bar{x}_k := x_k + \sum_{(0,0,l_3,l_4) \in \bigcup_{\frac{a}{2} \leq i < \frac{r_1}{2}} N^{[1]}_{r_2-r_1}(i)} c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ with some $c_{kl_3l_4} \in \mathbb{C}$.

(iii) There exists some $k = 1, 2$ with $\mathrm{mult}_E\, \bar{x}_k = r_1/2$ such that the monomials $\bar{x}_k$ and $x_3^{l_3} x_4^{l_4}$ for $(0, 0, l_3, l_4) \in N_{r_2-r_1}(r_1/2)$ form bases of $V(r_1/2)$. By permutation we may assume that $\bar{x}_k = \bar{x}_2$.

(iv) One of the following holds.

  (a) $\bar{x}_2 x_4$ and $x_3^{l_3} x_4^{l_4}$ for $(0, 0, l_3, l_4) \in N^{[1]}_{r_2-r_1}(r_1/2 + 1)$ are linearly independent in $V^{[1]}(r_1/2 + 1)$. In this case $\mathrm{mult}_E\, \hat{x}_1 = r_2/2$ for $\hat{x}_1 := \bar{x}_1 + \sum_{(0,l_2,l_3,l_4) \in \bigcup_{\frac{r_1}{2} \leq i < \frac{r_2}{2}} N^{[1]}_{r_2-r_1}(i)} c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ with some $c_{l_2l_3l_4} \in \mathbb{C}$. For $i < (r_1 + r_2)/2$, the monomials $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ for $\vec{l} \in N_{r_2-r_1}(i)$ form bases of $V(i)$.

  (b) $\bar{x}_2 x_4$ and $x_3^{l_3} x_4^{l_4}$ for $(0, 0, l_3, l_4) \in N^{[1]}_{r_2-r_1}(r_1/2 + 1)$ are linearly dependent in $V^{[1]}(r_1/2+1)$. They have one non-trivial relation of form $\psi := \bar{x}_2 x_4 + h(x_3, x_4) = 0$ for which the monomial $x_3^{(r_1+2)/a}$ appears in h with non-zero coefficient. In this case $r_2 = r_1 + 4$, $(r_1+2)/a$ is odd, $\mathrm{mult}_E\, \psi = r_2/2$ and $\mathrm{mult}_E\, \hat{x}_1 = r_1/2 + 1$ for $\hat{x}_1 := \bar{x}_1 + \sum_{(0,l_2,l_3,l_4) \in N^{[1]}_2(r_1/2)} c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ with

some $c_{l_2 l_3 l_4} \in \mathbb{C}$. For $i < (r_1 + r_2)/2$, there exists a natural exact sequence

$$0 \to \bigoplus_{\vec{l} \in \tilde{N}_2(i+1-r_2/2)} \mathbb{C} \cdot \hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \psi \to$$

$$\bigoplus_{\vec{l} \in \tilde{N}_2(i)} \mathbb{C} \cdot \hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \oplus \bigoplus_{(0,0,l_3,l_4) \in \tilde{N}_2(i-r_2/2)} \mathbb{C} \cdot x_3^{l_3} x_4^{l_4} \psi \to V(i) \to 0,$$

where $\tilde{N}_2(i) := \{\vec{l} \in \mathbb{Z}_{\geq 0}^4 \mid (r_1/2+1)l_1 + (r_1/2)l_2 + (a/2)l_3 + l_4 = i\}$.

Proof. $\dim V^{[0]}(1) = 1$, $\dim V^{[1]}(a/2) = 1$ and $\dim V^{[1]}(i) = 0$ for $i < a/2$ by Lemma 7.2 and $a < r_1 < r_2$ in Theorem 4.3(ii). This implies (i).

By permutation we may assume that $\mathrm{mult}_E x_3 = a/2$. The linear independence of x_3^2 and x_4^a is obtained by the method in Lemma 6.2. By $\dim V^{[0]}(1) = 1$, a general hyperplane section H_X on X has the defining function $x_4 + g_H = 0$ with $\mathrm{mult}_E g_H \geq 2$. Similarly by $\dim V^{[1]}(a/2) = 1$, a general elephant S_X of X has the defining function $x_3 + g_S = 0$ with $\mathrm{mult}_E g_S \geq a/2 + 1$. In Step 1 in Theorem 4.19 and Step 2 in Theorem 5.12, we have seen that we may choose the local coordinates x_{21}, x_{22}, x_{23} with weights $\mathrm{wt}(x_{21}, x_{22}, x_{23}) = (1, -1, b_2)$ of the index-one cover $Q_2^\sharp \in Y^\sharp$ so that the pre-images $H^\sharp$, $S^\sharp$ are given by respectively $x_{21} = 0$, $x_{23} = 0$. We let h denote a semi-invariant function with weight -1 on the germ $Q_2^\sharp \in Y^\sharp$ which defines E. Then on $Q_2 \in Y$ the functions x_3^2, x_4^a are expressed as $x_3^2 = u_3 x_{23}^2 h^a + p_3$ and $x_4^a = u_4 x_{21}^a h^a + p_4$, where u_3, u_4 are units and $p_3, p_4 \in (h^{a+1})$. Then, in order to obtain the linear independence of x_3^2, x_4^a, we need to show that h does not divide any function $c_1 x_{23}^2 + c_2 x_{21}^a$ for units c_1, c_2 with $(c_1, c_2) \neq (0, 0)$. This follows from $\mathrm{wt}\, x_{23}, \mathrm{wt}\, x_{23}^2 \not\equiv -1$ modulo r_2 by $b_2 = (a + r_2)/2$ with $a < r_1 \leq r_2 - 2$.

Because of the linear independence of x_3^2, x_4^a, for any $i \in \mathbb{Z}/2$ the monomials $x_3^{l_3} x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_{r_2-r_1}(i)$ are linearly independent in $V(i)$. For, every linear combination of the monomials $x_3^{l_3} x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_{r_2-r_1}^{[j]}(i)$ decomposes into a product of x_3^j, $x_4^{i-(j/2+\lfloor i/a-j/2 \rfloor)a}$ and $\lfloor i/a - j/2 \rfloor$ linear combinations of x_3^2, x_4^a. Set $W(i)$ as the linear subspace of $V(i)$ spanned by these monomials, and decompose it as $W(i) = W^{[0]}(i) \oplus W^{[1]}(i)$ by setting $W^{[j]}(i) := W(i) \cap V^{[j]}(i)$. Then $\dim W^{[j]}(i) = \#N_{r_2-r_1}^{[j]}(i)$ for $i < r_1/2$ and thus we obtain (ii) by Lemma 7.2.

The vector space $V(r_1/2)$ is spanned by $\bar{x}_1$, $\bar{x}_2$ and $W(r_1/2)$ by (ii). Since $\dim V(r_1/2) = \dim W(r_1/2) + 1$ by Lemma 7.2, we obtain (iii). By permutation we assume that $\bar{x}_2$ forms a basis of $V(r_1/2)/W(r_1/2) \cong \mathbb{C}$. Then $\mathrm{mult}_E x_1' \geq r_1/2 + 1$ for $x_1' := \bar{x}_1 + \sum_{(0,l_2,l_3,l_4) \in N_2^{[1]}(r_1/2)} c_{l_2 l_3 l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ with some $c_{l_2 l_3 l_4} \in \mathbb{C}$, and $V(r_1/2 + 1)$ is spanned by the monomials $x_1'^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ with $\vec{l} \in N_2(r_1/2 + 1)$. If $r_2 = r_1 + 2$ then by $\dim V(r_2/2) = \#N_2(r_2/2)$ in Lemma 7.2, the monomials above form bases of $V(r_2/2)$. Setting $\hat{x}_1 := x_1'$, we obtain the claim (iv)(a) for $i \leq r_2/2$. The rest of (iv)(a) follows from this and Lemma 7.2 inductively. If $r_2 = r_1 + 4$ then $\dim V^{[0]}(r_1/2 + 1) = \#N_4^{[0]}(r_1/2 + 1) = \#N_2^{[0]}(r_1/2 + 1)$ but $\dim V^{[1]}(r_1/2 + 1) = \#N_4^{[1]}(r_1/2 + 1)$

$1) = \#N_2^{[1]}(r_1/2 + 1) - 1$. Hence the monomials x_1', $\bar{x}_2 x_4$ and $x_3^{l_3} x_4^{l_4}$ in $W^{[1]}(r_1/2 + 1)$ have one non-trivial relation, say $\psi = 0$, in $V^{[1]}(r_1/2 + 1)$. If x_1' appears in ψ with non-zero coefficient, then $\mathrm{mult}_E \hat{x}_1 \geq r_2/2$ for $\hat{x}_1 := x_1' + \sum_{(0,l_2,l_3,l_4)\in N_4^{[1]}(r_1/2+1)} c_{l_2 l_3 l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ with some $c_{l_2 l_3 l_4} \in \mathbb{C}$, and eventually we obtain the case (iv)(a) as in the case $r_2 = r_1 + 2$. If x_1' does not appear in ψ, then by (iii) ψ is of form described in the statement (iv)(b) up to constant. Set $\hat{x}_1 := x_1'$. The vector space $V^{[1]}(r_2/2)$ is spanned by the function ψ and the monomials $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ for $\vec{l} \in N_2^{[1]}(r_2/2)$ with a relation $x_4 \psi = 0$ in $V^{[1]}(r_2/2)$. Since $\dim V^{[1]}(r_2/2) = \#N_4^{[1]}(r_2/2) = \#N_2^{[1]}(r_2/2)$ in Lemma 7.2, the function ψ does not vanish in $V^{[1]}(r_2/2)$. This means that $\mathrm{mult}_E \psi = r_2/2$. Now we obtain the claim (iv)(b) inductively, noticing the estimate $\dim V(i) = \#N_4(i)$ in Lemma 7.2 and the equality $\#N_4(i) = \#\tilde{N}_2(i) + \#\tilde{N}_2(i - r_2/2) - \#\tilde{N}_2(i + 1 - r_2/2)$ for $i < (r_1 + r_2)/2$ by direct calculation.   q.e.d.

We derive that P is of type $cD/2$ as an easy corollary. This completes Theorems 1.1(ii).

Corollary 7.4. *P is of type $cD/2$.*

Proof. P is of type $cAx/2$ or $cD/2$ by Table 3. Suppose that P is of type $cAx/2$. Then thanks to [24], we have an identification

$$P \in X \cong o \in (x_1^2 + x_4^2 + g(x_2, x_3) = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4} / \frac{1}{2}(1, 1, 1, 0)$$

with $g \in (x_2, x_3)^4$. $\mathrm{mult}_E x_4 = 1$ and $\mathrm{mult}_E x_i \geq a/2$ for any $i = 1, 2, 3$ by Lemma 7.3(i). Hence $a = 2$ and $\mathrm{mult}_E x_1 = 1$ by the relation $-x_4^2 = x_1^2 + g$ on X with $\mathrm{mult}_E(x_1^2 + g) \geq a$. Then $\mathrm{mult}_E(x_1^2 + x_4^2) = \mathrm{mult}_E g \geq 4$, which contradicts Lemma 7.3(ii).   q.e.d.

In the rest of this section we prove Theorem 1.2 by applying Lemma 6.1, as in Section 6 and [14, Section 6]. According to [24, Remark 23.1] by Mori, we can identify $P \in X$ with one of the following in the space $\mathbb{C}^4_{x_1 x_2 x_3 x_4} / \frac{1}{2}(1, 1, 1, 0)$.

(7.3) $o \in (x_1^2 + x_2 x_3 x_4 + x_2^{2\alpha} + x_3^{2\beta} + x_4^{\gamma} = 0),$

(7.4) $o \in (x_1^2 + x_2^2 x_4 + \lambda x_2 x_3^{2\alpha-1} + g(x_3^2, x_4) = 0),$

where $\alpha, \beta \geq 2$, $\gamma \geq 3$, $\lambda \in \mathbb{C}$ and $g \in (x_3^4, x_3^2 x_4^2, x_4^3)$. Note that

$$(7.5) \qquad \frac{r_1 + r_2}{2} \leq \begin{cases} \gamma & \text{in (7.3)}, \\ \mathrm{ord}\, g(0, x_4) & \text{in (7.4)}. \end{cases}$$

because a general elephant S_X is of type D_k with $k \geq r_1 + r_2$ by the dual graph of f_S in Theorem 4.3(ii). The crucial step is to obtain the expression of the germ $P \in X$ described in Theorem 1.2.

Lemma 7.5. *The germ $P \in X$ has the form described in either* (i) *or* (ii) *in Theorem 1.2 such that each $\mathrm{mult}_E x_i$ coincides with the weight of x_i distributed in Theorem 1.2.*

Proof. First consider the case (7.3). We prove that this case does not happen. By Lemma 7.3(i), $\mathrm{mult}_E x_4 = 1$ and $\mathrm{mult}_E x_i \geq a/2$ for any $i = 1, 2, 3$ with some $\mathrm{mult}_E x_i = a/2$. $\mathrm{mult}_E x_1 \geq a/2 + 1$ by the relation $-x_1^2 = x_2 x_3 x_4 + x_2^{2\alpha} + x_3^{2\beta} + x_4^{\gamma}$ and (7.5). Thus we may assume that $\mathrm{mult}_E x_3 = a/2$ without loss of generality. We construct $\bar{x}_1, \bar{x}_2$ as in Lemma 7.3(ii).

Set $W(i)$ as the linear subspace of $V(i)$ spanned by the monomials in x_3 and x_4, as in the proof of Lemma 7.3. If $\bar{x}_1 \notin W(r_1/2)$, then the triple $(\bar{x}_1, x_3, x_4)$ plays the role of $(\bar{x}_2, x_3, x_4)$ in Lemma 7.3(iii). We construct $\hat{x}_2$ as in Lemma 7.3(iv) and obtain the quartuple $(\hat{x}_2, \bar{x}_1, x_3, x_4)$. Set $\bar{x}_1 := x_1 + p_1(x_3, x_4)$ and $\hat{x}_2 := x_2 + p_2(\bar{x}_1, x_3, x_4)$. Then (7.3) is rewritten as

$$o \in ((\bar{x}_1 - p_1)^2 + (\hat{x}_2 - p_2)x_3 x_4 + (\hat{x}_2 - p_2)^{2\alpha} + x_3^{2\beta} + x_4^{\gamma} = 0).$$

Hence there exists a relation $\bar{x}_1^2 + (\text{something}) = 0$ in $V(r_1)$, which contradicts Lemma 7.3(iv). If $\bar{x}_1 \in W(r_1/2)$, then the triple $(\bar{x}_2, x_3, x_4)$ satisfies the condition in Lemma 7.3(iii). We construct $\hat{x}_1$ as in Lemma 7.3(iv), and set $\bar{x}_2 := x_2 + p_2(x_3, x_4)$ and $\hat{x}_1 := x_1 + p_1(\bar{x}_2, x_3, x_4)$. Note that the monomials $\bar{x}_2$, x_3 do not appear in p_1 by $\mathrm{mult}_E x_1 \geq a/2 + 1$ and the assumption $\bar{x}_1 \in W(r_1/2)$. Then (7.3) is rewritten as

$$o \in ((\hat{x}_1 - p_1)^2 + (\bar{x}_2 - p_2)x_3 x_4 + (\bar{x}_2 - p_2)^{2\alpha} + x_3^{2\beta} + x_4^{\gamma} = 0).$$

Hence there exists a relation $\bar{x}_2 x_3 x_4 + (\text{something}) = 0$ in $V((r_1 + a + 2)/2)$. Thus by Lemma 7.3(iv), the case (b) in Lemma 7.3(iv) holds and this relation must equal $(x_3 + (\text{something}))\psi = 0$. Then there exists a relation $x_3 \psi + (\text{something}) = 0$ in $V((r_1 + a + 4)/2)$, whence $(r_1 + a + 4)/2 \geq (r_1 + r_2)/2$ by Lemma 7.3(iv)(b). This inequality contradicts that $a < r_1 = r_2 - 4$.

We then consider the case (7.4). As in the case (7.3), we have $\mathrm{mult}_E x_4 = 1$ and $\mathrm{mult}_E x_1 \geq a/2 + 1$. Considering the relation $-x_2^2 x_4 = x_1^2 + \lambda x_2 x_3^{2\alpha - 1} + g$, we also have $\mathrm{mult}_E x_2 \geq a/2 + 1$. Thus $\mathrm{mult}_E x_3 = a/2$ by Lemma 7.3(i). Construct $\bar{x}_1, \bar{x}_2$ and $W(i)$ as in the case (7.3). The case $\bar{x}_1 \notin W(r_1/2)$ does not happen by the same reason as in the case (7.3), whence $\bar{x}_1 \in W(r_1/2)$. Construct $\hat{x}_1$, and set $\bar{x}_2 := x_2 + p_2(x_3, x_4)$ and $\hat{x}_1 := x_1 + p_1(\bar{x}_2, x_3, x_4)$ in the same way. The monomials $\bar{x}_2$, x_3 do not appear in p_1. Then (7.4) is rewritten as

$$(7.6) \quad o \in ((\hat{x}_1 - p_1)^2 + (\bar{x}_2 - p_2)^2 x_4 + \lambda(\bar{x}_2 - p_2)x_3^{2\alpha - 1} + g(x_3^2, x_4) = 0).$$

If $r_2 = r_1 + 2$, then Lemma 7.3(iv)(a) holds and the defining function in (7.6) must consist of the monomials $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ for $\vec{l} \in \bigcup_{i \geq (r_1 + r_2)/2} \tilde{N}_2(i)$. In particular $p_2 = 0$. p_1 is a function of x_3, x_4 by $\bar{x}_1 \in W(r_1/2)$. Hence (7.6) is written as in the case (i) of Theorem 1.2. The weighted homogeneous part of ϕ of weight $(r_1 + r_2)/2$ has multiplicity $> (r_1 + r_2)/2$ along E, whence it is irreducible by Lemma 7.3(iv)(a). If $r_2 = r_1 + 4$, then either (a) or (b) in Lemma 7.3(iv) holds. Since there exists a relation $\bar{x}_2^2 x_4 + (\text{something}) = 0$ in $V(r_1 + 1)$ by (7.6), the case (a) does not happen and in the case (b) this relation must equal $(x_2 + (\text{something}))\psi = 0$. Note that p_1 is a function of x_3, x_4 in the case (b). By replacing x_3 with its scalar multiple, we can set

$\psi = \bar{x}_2 x_4 + x_3^{(r_1+2)/a} + q(x_3^2, x_4)x_3 x_4$. Then (7.6) is rewritten as

$$o \in (\hat{x}_1^2 - 2\hat{x}_1 p_1 + (\bar{x}_2 - 2p_2 - qx_3)\psi$$
$$+ \bar{x}_2(\lambda x_3^{2\alpha-1} - x_3^{(r_1+2)/a}) + \tilde{g}(x_3^2, x_4) = 0).$$

By this expression, Lemma 7.3(iv)(b) implies that the weighted orders of p_1, $2p_2 + qx_3$, $\lambda x_3^{2\alpha-1} - x_3^{(r_1+2)/a}$, $\tilde{g}$ with weights $\mathrm{wt}(x_3, x_4) = (a/2, 1)$ are respectively $\geq r_1/2+1$, $\geq r_1/2$, $\geq r_2/2$, $\geq (r_1+r_2)/2$. Hence $p_1 = p_2 = 0$ and $\lambda x_3^{2\alpha-1} = x_3^{(r_1+2)/a}$. Replacing the coordinate $\bar{x}_2$ with $\bar{x}_2 - qx_3$ and setting $x_5 := -\psi$, we obtain the expression of $P \in X$ in Theorem 1.2(ii).   q.e.d.

It is the time to complete Theorem 1.2. By Lemma 7.5, we have the description of the germ $P \in X$ in either (i) or (ii) in Theorem 1.2. We take this description and apply Lemma 6.1. Of course we need to extend Lemma 6.1 in treating the case (ii) of Theorem 1.2, where $P \in X$ is given by a cyclic quotient of a complete intersection three-fold in $\mathbb{C}^5$, but this extension is straightforward. We let $g\colon (Z \supset F) \to (X \ni P)$ denote the weighted blow-up with weights $\mathrm{wt}\, x_i = \mathrm{mult}_E x_i$, induced from $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} := \mathbb{C}^4_{x_1 x_2 x_3 x_4}/\frac{1}{2}(1,1,1,0) \ni P)$ in the case of Theorem 1.2(i), and from $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} := \mathbb{C}^5_{x_1 x_2 x_3 x_4 x_5}/\frac{1}{2}(1,1,1,0,1) \ni P)$ in the case of Theorem 1.2(ii). By direct calculation as in the paragraph before Theorem 6.5, we verify the assumption of Lemma 6.1 and the implication $\mathrm{Sing}\, Z \subset \bigcup_i \bar{D}_i$. Therefore by Lemma 6.1, we obtain that f coincides with g over X. This completes Theorem 1.2.

Erratum to the previous paper

We correct the omission of the case (ii) of Theorem 4.3 and two minor gaps in the study of the Gorenstein case in the previous paper [14], by the methods used in this paper. First we state the additional case in the classification.

Addendum. *Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a three-fold divisorial contraction whose exceptional divisor E contracts to a Gorenstein point P of type cD. Then f can be of type $\mathrm{IIb}^{\vee\vee}$ in [14, Theorem 1.4] described below, in addition to the cases stated in [14, Theorem 1.8, Corollary 1.15].*

(i) *f belongs to the case (a) in Theorem 4.3(ii). There exists a suitable identification of $P \in X$ with*

$$o \in (\phi := x_1^2 + x_1 q(x_3, x_4) + x_2^2 x_4 + \lambda x_2 x_3^2 + \mu x_3^3 + p(x_2, x_3, x_4) = 0)$$

in the space $\mathbb{C}^4_{x_1 x_2 x_3 x_4}$ with $p \in (x_2, x_3, x_4)^4$, such that f is the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r+1, r, a, 1)$ which satisfies the following conditions.
(a) *$a \mid 2r+1$ with $a \neq 1, 2r+1$.*
(b) *ϕ has weighted order $2r+1$ with the weights distributed above. Its weighted homogeneous part of weight $2r+1$ is irreducible. In fact $x_1 q$ has weighted order $2r+1$ unless $q = 0$.*

(ii) f *belongs to the case* (b) *in Theorem* 4.3(ii). *There exists a suitable identification of* $P \in X$ *with*

$$o \in \begin{pmatrix} \phi_1 := x_1^2 + x_2 x_5 + p(x_2, x_3, x_4) = 0 \\ \phi_2 := x_2 x_4 + x_3^{(r+1)/a} + q(x_3, x_4)x_4 + x_5 = 0 \end{pmatrix}$$

in the space $\mathbb{C}^5_{x_1 x_2 x_3 x_4 x_5}$ *with* $p \in (x_2, x_3, x_4)^4$, *such that* f *is the weighted blow-up with weights* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (r+1, r, a, 1, r+2)$ *which satisfies the following conditions.*

  (a) $a \mid r + 1$ *with* $a \neq 1, r + 1$.
  (b) ϕ_1 *has weighted order* $2r + 2$ *with the weights distributed above. Similarly* $q x_4$ *has weighted order* $r + 1$ *unless* $q = 0$.

There exist examples of this addendum.

Example.   (i) Let $P \in X$ be the germ

$$o \in (x_1^2 + x_2^2 x_4 + x_3^{(2r+1)/a} + x_4^{2r+1} = 0) \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4},$$

where $a \mid 2r + 1$ with $a \neq 1, 2r + 1$. P is of type cD. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r + 1, r, a, 1)$. Then f is a divisorial contraction with discrepancy a. f is of type $\mathrm{IIb}^{\vee\vee}$ with $J = \{(r, 1), (r + 1, 1)\}$.

(ii) Let $P \in X$ be the germ

$$o \in \begin{pmatrix} x_1^2 + x_2 x_5 + x_4^{2r+2} = 0 \\ x_2 x_4 + x_3^{(r+1)/a} + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^4_{x_1 x_2 x_3 x_4 x_5},$$

where $a \mid r + 1$ with $a \neq 1, r + 1$. P is of type cD. Let f be the weighted blow-up of X with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (r + 1, r, a, 1, r + 2)$. Then f is a divisorial contraction with discrepancy a. f is of type $\mathrm{IIb}^{\vee\vee}$ with $J = \{(r, 1), (r + 2, 1)\}$.

The omission is due to an error in the proof of [14, Lemma 3.14]. The triple $(x, y, z) = (1, 0, 2)$ also satisfies the equality $(x - 1 + y)/r_1 = (x + 1 - z)/r_2$ appearing in the proof, and the addendum is derived from this case. We first deduce Theorem 4.3(ii), and then obtain the description of f by using Theorem 4.3(ii).

Theorem 4.3(ii) in this case is proved in the same fashion as Theorem 4.19. However we need to obtain the general elephant theorem in the course of the proof, because Theorem 4.14 does not hold in the Gorenstein case. The proof is modified as follows. We first see the statement on $Q_1^\sharp$ in Step 1 in Theorem 4.19, from which Step 3 in Theorem 4.19 is deduced. We then see that $H^\sharp$ is smooth at $Q_2^\sharp$. This is the case unless $V \cong \mathbb{P}^1$ with $a = 2$ or $(r_2 + 1)/2$ by $(H \cdot V)_{Q_2} = 2/r_2$. If $a = 2$ with $V \cong \mathbb{P}^1$, then there exist new semi-invariant coordinates x_1, x_2, x_3 at $Q_2^\sharp$ with $(x_1, x_2, x_3)|_{V^\dagger} = (t^{2/r_2}, t^{1-2/r_2}, 0)$, whence $H^\sharp$ is smooth at $Q_2^\sharp$ by $(H \cdot V)_{Q_2} = 2/r_2$. If $a = (r_2 + 1)/2$, then $(a, r_1, r_2) = (3, 4, 5)$ and $[H \cap E] = 3[C]$ by Step 3. $E^\sharp$ is not smooth at $Q_2^\sharp$ by $(x_{21}, x_{22}, x_{23})|_{C^\dagger} = (0, t^{1/r_2}, 0)$ and $C \subset E$, whence $H^\sharp$ is smooth at $Q_2^\sharp$ by $(H \cdot E \cdot [x_{22} = 0])_{Q_2} = 3/5$. Thus $H^\sharp$ is smooth at $Q_2^\sharp$, and Step 2 in Theorem 4.19 is deduced. This implies Step 1 in Theorem 4.19. We also

obtain the general elephant theorem since $|-K_Y|$ is free outside C. Hence Theorem 4.3(ii) in this case is proved.

S_X has a Du Val singularity of type D at P, whence P is cA or cD. The classification in the cA case in [14, Section 6] works even if this extra case is taken into account, because we do not need the result of [14, Section 5] in the argument before [14, Lemma 6.6] whenever $a < r_1$. Hence we obtain that P is cD in the case (ii) of Theorem 4.3. Now Addendum is proved in the same way as in Section 7. First obtain the identification of $P \in X$ with $o \in ((\bar{x}_1 - p_1(x_3, x_4))^2 + g(x_2, x_3, x_4) = 0)$ with $g \in (x_2, x_3, x_4)^3$ and $\mathrm{mult}_E\, \bar{x}_1, \mathrm{mult}_E\, x_2 \geq r_1$, $\mathrm{mult}_E\, x_3 = a$, $\mathrm{mult}_E\, x_4 = 1$. The pivotal remark is that g contains one of the monomials $x_2^2 x_4, x_2 x_3 x_4, x_3^2 x_4, x_2 x_4^2, x_3 x_4^2, x_4^3$ since S_X has a Du Val singularity of type D at P. Then the rest of the proof is similar.

Below we fill two minor gaps in [14]. Firstly we correct the proof of $c_2 = 1$ in [14, Lemma 3.11(i)]. Its proof implies only that $\overline{ac_2} = ac_2$. Suppose that $c_2 \geq 2$. By the argument in Step 3 in Lemma 4.12, we obtain that $a = 2$, $r_1 = 3$ and r_2 is an odd integer $> 3c_2$. We then compute the length of the cokernel of the map [14, (2.7)] at Q_2. This is $c_2 + \min\{0, (r_2 + 1)/2 - 2c_2\}$ for $c_2 < r_2/3$, and is bounded by two by [14, Lemma 2.10]. Hence $c_2 = 2$. By $(H \cdot C) = 1/3 - 2/r_2$ and $(H \cdot H \cdot E) = (1/3 + 1/r_2)/2$, we have $[H \cap E] = \alpha[C] + \beta[C']$ with $(r_2, \alpha, \beta) = (7, 5, 0)$, $(7, 2, 1)$, $(9, 2, 0)$, $(9, 1, 1)$ or $(15, 1, 0)$, where $C' \cong \mathbb{P}^1$ is a curve intersecting C at Q_2. $H^\sharp$ is not smooth at $Q_1^\sharp$ by $(x_{11}, x_{12}, x_{13})|_{C^\dagger} = (t^{1/3}, 0, 0)$ and $C \subset H$, whence $\alpha \neq 1$. If $(r_2, \alpha, \beta) = (7, 2, 1)$ then $(x_{21}, x_{22}, x_{23})|_{C^\dagger} = (0, t^{2/7}, t^{3/7})$ and $C'^\dagger$ is given by x'_{21}-axis for new coordinates $x'_{21}, x'_{22}, x'_{23}$. For a suitable surface T on $Q_2 \in Y$ given by $x_{21}^2 - cx_{23} = 0$ with $c \in \mathbb{C}$, T intersects C, C' properly with $(T \cdot H \cdot E)_{Q_2} = (T \cdot 2C + C')_{Q_2} = 8/7$. This contradicts that $H^\sharp$, $E^\sharp$ at $Q_2^\sharp$ have multiplicities ≥ 3 by $C, C' \subset H, E$. Hence $(r_2, \alpha, \beta) = (7, 5, 0)$ or $(9, 2, 0)$. We obtain the coordinates x_{i1}, x_{i2}, x_{i3} of $Q_i^\sharp \in Y^\sharp$ such that $Q_1^\sharp \in E^\sharp$ is given by $x_{12} = 0$ and $Q_2^\sharp \in H^\sharp$ is given by $x_{21} = 0$. Take a general elephant S. By [14, Lemma 3.4] we have $[S \cap E] = \gamma[C] + [D_\delta]$ with $(r_2, \gamma, \delta) = (7, 3, 1)$, $(7, 4, 2)$, $(7, 1, 2)$, $(9, 1, 1)$ or $(9, 1, 2)$, where D_δ is a union of $\mathbb{P}^1$ intersecting C at Q_δ. We can remove the cases $(r_2, \gamma, \delta) = (7, 1, 2)$ and $(9, 1, 2)$ using a surface T on $Q_2 \in Y$ given by $x_{21} - cx_{22}x_{23} = 0$ with $c \in \mathbb{C}$. If $\delta = 1$ then $D_1 \cong \mathbb{P}^1$, $(H \cdot D_1)_{Q_1} = 1/3$ and $D_1 \not\subseteq H$. This contradicts that $H^\sharp$ is not smooth at $Q_1^\sharp$. Since S is a general element of $|2H|$, the defining function of $S^\sharp$ at $Q_2^\sharp$ contains the monomial x_{21}^2 with non-zero coefficient. In the case $(r_2, \gamma, \delta) = (7, 4, 2)$, this expression contradicts that $C, D \subset S$.

Secondly the smoothness of E at the generic point of C is indispensable in the argument after the equation (3.6) in [14, Theorem 3.15]. Set $[S \cap E] = x[C] + [D]$ as in the proof. We can express $E^\sharp$ on $Q_2^\sharp \in Y^\sharp$ as $x_{23} \cdot$ (something) $+ x_{21}^x y = 0$, where y is the semi-invariant function with weight -1 for which $D^\sharp$ is defined by $x_{23} = y = 0$. In particular we have $r_2 \mid x$ by considering weights. Then $r_2 = r_1 + 1$ by (3.6) in [14, Theorem 3.15], $a < r_1 < r_2$ and $[H \cap S] = [C] + [B]$ with $(E \cdot B)_{Q_1} = 2/r_1$. Thus $E^\sharp$ at $Q_1^\sharp$ is smooth unless $a = r_1 - 2$ or $(r_1 - 1)/2$. If $a = r_1 - 2$ or $(r_1 - 1)/2$ then by $a \mid r_1 + r_2$ eventually $(r_1, r_2) = (7, 8)$ and $(a, u) = (5, 3)$ or $(3, 5)$ with

$[H \cap E] = u[C]$. Hence $E^\sharp$ at $Q_1^\sharp$ is smooth by $(H \cdot E \cdot [x_{11} = 0])_{Q_1} = u/7$, and thus E is smooth at the generic point of C.

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